ANALYTICAL AND NUMERICAL THEORY
OF THE MOTIONS OF THE ORBITAL PLANES OF
JUPITER'S SATELLITES.
SECULAR TERMS.

A. J. LECKIE.

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SECULAR TERMS.

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Aan de nagedachtenis van mijn vader.

Aan mijn vrouw.

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ANALYTICAL AND NUMERICAL THEORY OF THE MOTIONS OF THE ORBITAL PLANES OF JUPITER'S SATELLITES.

INTRODUCTION.

If the motions of the four large satellites of Jupiter are referred to a fixed fundamental plane, e. g. the orbital plane of Jupiter at the zero epoch, and if we introduce, as is customary, the variables:

$$p_i = \sin i_i \sin \theta_i$$
$$q_i = \sin i_i \cos \theta_i$$

where i_i is the inclination of a satellite i to the fundamental plane and θ_i is the longitude of the node of the same satellite, counted from a fixed point in the fundamental plane along it, — it is well known that analytical expressions can be derived for p_i and q_i, which, if we take into account secular inequalities only, are of the form:

$$p_i = \Sigma_j \sigma_{ij} N_j \cos (b_j \tau + M_j)$$
$$q_i = \Sigma_j \sigma_{ij} N_j \sin (b_j \tau + M_j)$$

where σ_{ij} and b_j are constants depending on the masses and other elements. In 1887 Souillart published his second memoir on the theory of Jupiter's satellites, a numerical evaluation of his former analytical theory. Among the fundamental constants occurring in this paper are the constants b_j and σ_{ij}, derived by Souillart from the integration of his differential equations for p_i and q_i by two different methods. It might be expected that the values of b_i and σ_{ij}, as computed by both methods, would agree. This, however, is not the case. As prof. De Sitter has pointed out in his "*Determination of the inclinations and nodes of the orbits of Jupiter's satellites*" the theoretical constants b_i and σ_{ij} as determined by Souillart according to his two methods of integration from the same masses and elements, differ in several cases by amounts, which, in their effect on the resulting values of p_i and q_i, can not be neglected, and are quite comparable with the errors of observation (p. 118). In the same paper Prof. de Sitter suggested that this defect of the theory can be remedied by using as the fundamental plane of the theory Jupiter's equator instead of its orbital plane. In a later publication "*On the libration of the three inner large satellites of Jupiter*" he returned to the same question and from his discussions drew the general conclusion that "Souillart's theory no longer satisfies our needs and that a satisfactory theory can only be made by starting from an "orbite intermédiaire" which includes the effect of the compression of the planet and of the commensurability of the mean motions of the satellites, while

neglecting on the other hand the own eccentricities and inclinations. Moreover the plane of Jupiter's equator should be used as the fundamental plane instead of its orbit".

The leading ideas of a new theory of the longitudes, starting from a first approximation which nearly coincides with the periodic orbit and which will make it possible to determine the motions of the satellites with a much higher degree of accuracy than was possible by the old method, as used by Souillart, is given by prof. de Sitter in Part I of this Volume. *)

In the theory of the inclinations and nodes however, the approximate mutual commensurability of the mean motions does not cause great difficulties and consequently for this part of the theory an elliptical orbit is quite sufficient as a first approximation.

The chief aim of a new theory of the inclinations and nodes therefore must consist in determining the fundamental plane so as to make not only the inclinations of the four satellites to this plane, but also that of Jupiter's equator small throughout. Such a plane is the plane of Jupiter's mean equator at the zero epoch. In the following paper the differential equations of a satellite have first been derived with respect to a fixed fundamental plane in the immediate neighbourhood of the mean equator, and afterwards transformed to the moving true equator of Jupiter. If the differential equations are then integrated according to both methods of Souillart, the differences that occurred in his theory seem to disappear; in order to decide on this point a numerical evaluation of the constants b_i and σ_{ij} according to both methods must be made.

Even if we disregard the difficulties already mentioned there are still other objections against Souillart's theory. Souillart had derived the values of the constants b_j and σ_{ij} with a degree of accuracy that is far beyond what can be reached by the observations. As can easily be seen, however, his values are not entirely trustworthy and that in more than one respect:

In the first place the numerical data used for the masses and the constant Jb^2, which depends on the compression of the planet, need large corrections (see e. g. de Sitter, Proc. Royal Ac. Amsterdam X pp. 718, 719), and consequently the theoretical constants that involve the elements must also be corrected. In order to find these corrections the developments of b_j and σ_{ij} in terms of those of the masses and other elements can be used (Cf. loc. cit. pp. 722, 723). These formulae show that, in consequence of the large corrections that must be added to Souillart's elements many of the decimal places of the b_j and σ_{ij}, as given by him, must be rejected as not trustworthy.

In the second place it is questionable, whether — even accepting the masses and other elements, used by Souillart as accurately known, — it is permissible to neglect the corrections contributed to the b_j by the part of the perturbative function that contains the terms of the fourth degree in the inclinations. Souillart has not investigated the influence of this part of the perturbative function (in this paper denoted by R^{IV}), but has assumed the corrections of the b_i due to R^{IV} as entirely negligible compared with the accuracy with which he wished to determine the b_i. In one of the following chapters the corrections Δb_i arising from R^{IV} have been derived and it is found that, if the b_i are computed to more than nine decimals, R^{IV} cannot be neglected; its influence on b_2 is pretty considerable and affects the eighth decimal place. Now the values of p_i and q_i can be derived from observations in 5 places, whence — as will be pointed out — the b_i and σ_{ij} must be known in 9 and 4 decimal places. Hence with this degree of accuracy R^{IV} must be taken into account. The influence of R^{IV} in Souillart's theory is not smaller, but considerably larger; thus, when computing his b_i in ten and more decimal places (expressed in degrees, while our b_i are expressed in radians), Souillart ought to have taken R^{IV} into account.

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*) Outlines of a new mathematical theory of Jupiter's satellites. Annalen van de Sterrewacht te Leiden, Deel XII, Eerste Stuk.

Lastly, there is still another point that has not been thoroughly treated by Souillart. The potential function of Jupiter, as employed by Souillart, is the same as Laplace had used in his theory. Hence Souillart neglects the terms containing Kb^4, etc. Now the constant K is imperfectly known, but its influence, although it is small, may not be entirely neglected. In adopting without any correction the potential function of Laplace — which completely satisfied the accuracy required by the latter, — Souillart has on the one hand taken small quantities into account, whilst on the other hand neglecting terms of the same amount.

The following paper is an endeavour to avoid some of the objections to Souillart's theory. In the first chapter the differential equations of the inclinations and nodes are derived. As will be seen, the elements used are not the common elliptical elements. The reason for using new elements instead of the keplerian is there given together with some other points in which our work differs from Souillart's.

The second chapter contains the development of the perturbative function in terms of the inclinations and nodes. The chief aim of this development consisted in determining the secular parts of the perturbative function, but the most important periodical and critical terms have also been developed. The method of developing the principal part — the reciprocal of the mutual distance of two satellites, or of a satellite and the sun — is quite different from the usual method of Leverrier, which was also used by Souillart. The principles of the new development have been given by Hill in a paper to be quoted further on. In the periodic parts of the perturbative function all terms of higher order than the second in the inclinations have been omitted, while the development of the secular terms is complete to the fourth powers in the inclinations.

The first part of the third chapter contains the analytical integration of the differential equations according to the method of variation of arbitrary constants, but in a completer form than Souillart had employed it. In the second part of this chapter the theoretical constants occurring in the analytical expressions have for the greater part been numerically evaluated by starting from a set of masses and other elements, that are certainly much nearer the truth than those chosen by Souillart. Pains have been taken to secure the values here computed — accepting the fundamental constants (masses, etc.) as accurately known — being entirely trustworthy.

In the fourth chapter the corrections have been developed to which the constants b_j and σ_{ij} are subject, if the elements be altered by small amounts. In reality the corrections to the elements may not be very small; it was therefore necessary to extend the developments and also take account of squares and in a few cases of higher powers of the corrections. These developments are very important since only by them it was possible to investigate whether, in order to reach the required accuracy, we have to use the perturbative function in such a complete form as that employed throughout the following; at the same time they enable us to make an estimate of the amount contributed to the values of b_i and σ_{ij} by each of the masses and other elements. Moreover these formulae can serve as a basis for determining a better set of values of the elements, especially of the masses.

In the next chapter the method and results of Souillart have been compared with ours and an attempt has been made to account for the small amounts by which the two systems of results disagree.

The last chapter contains an analytical development of the corrections Δb_i, that are due to the part R^{IV} of the perturbative function. They depend on the masses and other elements, but also on the quantities N_i, which denote the own inclinations, that are to be determined from observations. Adopting the expressions of the inclinations and nodes, as given by de Sitter in Proceedings Amsterdam X pp. 722, 723, 594 it was possible to derive new values of N_i and so make an estimate of the amount of Δb_i.

Throughout this paper frequent use has been made of the following papers which will be quoted as shown in brackets:

Souillart Théorie analytique des mouvements des Satellites de Jupiter, Memoirs of the Royal Astronomical Society Vol XLV, 1880 [*Souillart I*]

— Théorie analytique des mouvements des Satellites de Jupiter, seconde partie, réduction des formules en nombres, Mémoires présentés par divers savants à l'Académie des Sciences de l'Institut de France, 1889 [*Souillart II*]

de Sitter On the libration of the three inner large satellites of Jupiter, Publications of the Astronomical laboratory at Groningen, Vol 17. Groningen, Hoitsema Brothers [*Gron. publ.* 17]

— A determination of the inclinations and nodes of the orbits of Jupiter's satellites, Annals of the Cape Observatory, Vol. XII, part III [*Cape XII*, 3]

— A determination of the mass of Jupiter and elements of the orbits of its satellites, Annals of the Cape Observatory, Vol. XII, part I [*Cape XII*, 1]

— On the elements and masses of the satellites of Jupiter and the mass of the system, Proc. Royal. Ac. Amsterdam Vol. X [*Elements and masses*]

— Outlines of a new mathematical theory of Jupiter's satellites, Annalen van de Sterrewacht te Leiden, Deel XII, eerste stuk [*Leiden XII*, 1]

— Derivation of final inclinations and nodes of the orbital planes of Jupiter's satellites, Annals of the Cape Observatory Vol. XII, part V [*Cape XII*, 5]

CHAPTER I.

DERIVATION OF THE DIFFERENTIAL EQUATIONS.

In investigating the motion of Jupiter's satellites with respect to Jupiter itself, it is not permissible — as is commonly done in Planetary Astronomy — to neglect the dimensions of Jupiter and regard it as a material point, because on the one hand the dimensions of Jupiter are quite comparable with the distances of the satellites and on the other hand the compression of this planet is very considerable. For these reasons the potential function of Jupiter's attraction on a satellite cannot be identified with the quantity $f\frac{M}{r}$, where f designates the mutual attraction of two units of mass, M the mass of Jupiter and r its distance from the satellite. We need, therefore, a more complete form of this function.

The potential of an ellipsoid of revolution in a point outside the surface is: *)

$$U = f\left[\frac{M}{r} - \frac{4\pi\left(\frac{3}{2}\mu^2 - \frac{1}{2}\right)}{3.5.r^3}\int_0^b \rho\,\frac{d\left(b^5 e^2\sqrt{1-e^2}\right)}{db}\,db\right.$$
$$+ \frac{4\pi\left(\frac{35}{8}\mu^4 - \frac{15}{4}\mu^2 + \frac{3}{8}\right)}{5.7.r^5}\int_0^b \rho\,\frac{d\left(b^7 e^4\sqrt{1-e^2}\right)}{db}\,db$$
$$\left. - \frac{4\pi\left(\frac{231}{16}\mu^6 - \frac{315}{16}\mu^4 + \frac{105}{16}\mu^2 - \frac{5}{16}\right)}{7.9.r^7}\int_0^b \rho\,\frac{d\left(b^9 e^6\sqrt{1-e^2}\right)}{db}\,db + \ldots\right]$$

where: f is the Gaussian constant of attraction,

ρ is the density of the ellipsoid, supposed to be a continuous function of b,

$\mu = \cos\theta$, if θ is the angle between the radius-vector of the point and the axis of the ellipsoid,

r is the radius-vector of the point considered,

$2b$ is the equatorial diameter of the surface, $2a$ the axis of revolution,

M is the total mass of the ellipsoid.

Assuming the form of Jupiter to be an ellipsoid of revolution, (in reality this is not the case as has been shown by de Sitter in his paper *"On the form of the planet Jupiter"* Proc. Amst. 1915

*) Cf. Tisserand Méc. Cél. II p. 322.

p. 1056, but the deviation from the ellipsoid may be neglected for our problem), we find for the potential function with respect to a satellite of mass m_i and jovicentric latitude s_i — taking account of the relations $b^2 - a^2 = b^2\varepsilon^2$, $\sqrt{1-\varepsilon^2} = \frac{a}{b}$, $\cos\theta_i = \sin s_i$ —

$$U_i = fm_i \left[\frac{M}{r_i} + \frac{2\pi}{3.5}\frac{1}{r_i^3}(1 - 3\sin^2 s_i)\int_0^b \rho\,\frac{d[(b^2-a^2)\,b^2a]}{db}\,db + \right.$$
$$\left. + \frac{3\pi}{10.7}\frac{1}{r_i^5}(1 - 10\sin^2 s_i - \frac{35}{3}\sin^4 s_i)\int_0^b \rho\,\frac{d[(b^2-a^2)^2\,b^2a]}{db}\,db + \ldots\right]$$

If now we put:

$$MJb^2 = \frac{2}{5}\pi\int_0^b \rho\,\frac{d[(b^2-a^2)\,b^2a]}{db}\,db$$
$$MKb^4 = \frac{3}{7}\pi\int_0^b \rho\,\frac{d[(b^2-a^2)^2\,b^2a]}{db}\,db$$

we finally have:

$$U_i = fm_i\left[M\left\{\frac{1}{r_i} + \frac{Jb^2}{3r_i^3}(1 - 3\sin^2 s_i) + \frac{Kb^4}{10r_i^5}(1 - 10\sin^2 s_i + \frac{35}{3}\sin^4 s_i)\right\}\right].$$

The constants J and K, above expressed by definite integrals, thus depend on the inner distribution of the mass. The influence of K on the motion of the large satellites is too small to allow a determination of K from them. But the motion of the perijove and the node of the fifth satellite give a determination of K, if J is known from the motions of the large satellites.

Let a fixed system of coordinate axes with a fixed origin be chosen and let the coordinates of a body referred to these axes be denoted by x, y, z.

We shall consider the case that besides Jupiter there are n heavenly bodies (satellites of Jupiter, the Sun, Saturn, etc.) which we shall regard as material points. The differential equations of motion of each of these bodies with mass m_i and coordinates x_i, y_i, z_i then are:

(1) $$m_i\frac{d^2x_i}{dt^2} = \frac{\partial U}{\partial x_i}\,,\quad m_i\frac{d^2y_i}{dt^2} = \frac{\partial U}{\partial y_i}\,,\quad m_i\frac{d^2z_i}{dt^2} = \frac{\partial U}{\partial z_i}\,.$$

The value of U is:

$$U = \Sigma_i\, U_i + \sum_{i\neq j} m_i\, m_j\, U_{ij}$$

where the sums are to be taken over all values of i and j, excepting 0, and where:

$$U_i = fm_0\, m_i\left[\frac{1}{r_i} + \frac{1}{3}\frac{Jb^2}{r_i^3}(1 - 3\sin^2 s_i) + \frac{1}{10}\frac{Kb^4}{r_i^5}(1 - 10\sin^2 s_i + \frac{35}{3}\sin^4 s_i)\right]$$
$$U_{ij} = \frac{f}{\Delta_{ij}}$$

m_0 = mass of Jupiter

Δ_{ij} = distance between two bodies i and j.

If now we introduce relative coordinates x_i, y_i, z_i, thus:

$$x_i = \mathrm{x}_i - \mathrm{x}_\circ \quad , \quad y_i = \mathrm{y}_i - \mathrm{y}_\circ \quad , \quad z_i = \mathrm{z}_i - \mathrm{z}_\circ$$

where $\mathrm{x}_\circ$, $\mathrm{y}_\circ$, $\mathrm{z}_\circ$ are the coordinates of the centre of gravity of Jupiter, the differential equations become: *)

$$\frac{d^2x_i}{dt^2} = \frac{1}{m_i}\frac{\partial U}{\partial \mathrm{x}_i} - \frac{1}{m_\circ}\frac{\partial U}{\partial \mathrm{x}_\circ} \text{ , etc.}$$

These equations can be written in the form:

$$\frac{d^2x_i}{dt^2} = \frac{\partial \Omega_i}{\partial x_i} \quad , \quad \frac{d^2y_i}{dt^2} = \frac{\partial \Omega_i}{\partial y_i} \quad , \quad \frac{d^2z_i}{dt^2} = \frac{\partial \Omega_i}{\partial z_i}$$

where Ω_i is the force-function, the expression of which is:

$$\Omega_i = f\,(m_\circ + m_i)\left\{\frac{1}{r_i} + \frac{1}{3}\frac{Jb^2}{r_i^3}\,(1 - 3\,\sin^2 s_i) + \frac{1}{10}\frac{Kb^4}{r_i^5}\,(1 - 10\,\sin^2 s_i + \frac{35}{3}\,\sin^4 s_i)\right\}$$
$$+ fm_\circ \Sigma_j' m_j\left\{\frac{1}{\Delta_{ij}} - \frac{x_ix_j + y_iy_j + z_iz_j}{r_j^3}\left[1 + \frac{Jb^2}{r_j^2}\,(1 - 5\,\sin^2 s_j) + \frac{Kb^4}{r_j^4}\left(\frac{1}{2} - 7\,\sin^2 s_j + \frac{21}{2}\,\sin^4 s_j\right)\right]\right.$$
$$\left. - \frac{2\,(\alpha x_i + \beta y_i + \gamma z_i)}{r_j^2}\left[\frac{Jb^2}{r_j^2}\,\sin s_j + \frac{Kb^4}{r_j^4}\left(\sin s_j - \frac{7}{3}\,\sin^3 s_j\right)\right]\right\},$$

where Σ_j' denotes a sum over the values $j = 1, \ldots 4$, and the Sun, except however $j = i$.

As can be easily seen the part of Ω_i contributed by Saturn can be neglected in consequence of its large distance and relatively small mass, but the influence of Saturn on the Sun's coordinates must be taken into account.

The above expression of Ω_i is much more complicated than that used by Souillart. The main difference is that terms multiplied by the small factors J and K in the complementary part of the perturbative function have been omitted by Souillart and his predecessors. As will be shown afterwards this is only permissible if we disregard all but the secular terms.

In the theory of Jupiter's satellites it is convenient to define the motions of the satellites by a set of equations different to (1). Before deriving this, some particulars as to the choice of the plane of (x, y) may be given. As has been said in the Introduction, the fundamental plane must be chosen in such a way that the inclinations of the satellites to that plane are small. The fundamental plane adopted by us throughout this paper is a plane in the immediate neighbourhood of the mean position of Jupiter's equator and subject to the condition that at the zero epoch it co-incides with it.

We denote the longitude of the ascending node of Jupiter's equator by $180° - \psi$, reckoned along the plane of (x, y) and by π the inclination of that equator to the same plane, and we put:

$$\alpha = \sin\psi\,\sin\pi$$
$$\beta = \cos\psi\,\sin\pi$$
$$\gamma = \cos\pi$$

*) Cf. de Sitter, Leiden, XII, 1, p. 11.

By this choice of the plane of (x,y) the quantities ψ and π are very small and thus also α, β, while γ is almost equal to 1. The latitudes s_i are also very small. For the satellites the largest is smaller than $0°.7$, while even for the other bodies outside the Jupiter system they are small, for the Sun it does not exceed $3°.1$.

If now we put:

$$x_i' = \frac{dx_i}{dt} \quad , \quad y_i' = \frac{dy_i}{dt} \quad , \quad z_i' = \frac{dz_i}{dt}$$

and:

$$T_i = \tfrac{1}{2}\,(x_i'^2 + y_i'^2 + z_i'^2)$$
$$F_i = T_i - \Omega_i$$

the equations take the ordinary canonical form:

$$\frac{dx_i}{dt} = \frac{\partial F_i}{\partial x_i'} \quad , \quad \frac{dx_i'}{dt} = -\frac{\partial F_i}{\partial x_i} \quad , \text{ etc.}$$

Instead of the elliptical elements used by Souillart we now introduce a canonical set. The elements chosen are a slight modification of those proposed by Hill in his paper "*Motion of a system of material points under the action of gravitation*" (Astronomical Journal: N°s 646—647, April 1913). In a paper entitled "*On canonical elements*" (Proc. Amst. Vol. XXII p. 353) prof. de Sitter showed, how these elements could be derived in a much shorter and at the same time more general way than had been done by Hill.

One of the canonical systems derived in the paper just mentioned is the set:

$$v \quad , \quad g \quad , \quad \omega \quad , \quad V \quad , \quad G \quad , \quad \Omega$$

where:

g = the angular distance of the satellite's perijove from its node
ω = longitude of the node of the satellite's orbit
v = true anomaly
G = the constant of areas
$\Omega = G \cos i$, where i is the satellite's inclination.

Applying to this system of variables the canonical transformation:

$$f = v \qquad w = v + g + \omega \text{ (true orbit longitude)} \qquad \psi = -\omega$$
$$F = V - G \qquad W = G \qquad \Psi = G - \Omega$$

we obtain the canonical set:

$$\frac{df_i}{dt} = \frac{\partial H_i}{\partial F_i} \qquad \frac{dw_i}{dt} = \frac{\partial H_i}{\partial W_i} \qquad \frac{d\psi_i}{dt} = \frac{\partial H_i}{\partial \Psi_i}$$

$$\frac{dF_i}{dt} = -\frac{\partial H_i}{\partial f_i} \qquad \frac{dW_i}{dt} = -\frac{\partial H_i}{\partial w_i} \qquad \frac{d\Psi_i}{dt} = -\frac{\partial H_i}{\partial \psi_i}$$

$$H_i = T_i - \Omega_i$$

where:

H_i is the energy function,
T_i the kinetic energy,
Ω_i the force function.

We denote the last set of canonical elements as "isoprotometric elements", a name given to them by de Sitter because the semidiameter a of the intermediate ellipse, called "protometer" by Gylden, is constant. The use of these elements offers a great advantage over that of the intermediate keplerian ellipse as used in the classical method of variation of constants, for now we have at our disposal the constant a and may define it so that the period in the intermediate orbit is the same as in the actual orbit, which is very desirable. By this condition the semidiameter of the satellite's orbit is accurately defined; the numerical values are slightly different from those of Souillart who has included in the values of a, as given by the third law of Kepler, perturbations of the first order (this, however, is not the original meaning of Lagrange's method of variation of constants, but an extension of it, in order to get a better intermediate orbit to start from.)*

For the investigation of the inclinations and nodes we need the equations:

$$\text{(A)} \qquad \frac{d\psi_i}{dt} = \frac{\partial H_i}{\partial \Psi_i} \quad , \quad \frac{d\Psi_i}{dt} = -\frac{\partial H_i}{\partial \psi_i} \, .$$

The expression for the kinetic energy is:

$$T_i = \frac{\beta_i^2}{2}\left[\frac{2}{r_i} - \frac{1}{a_i}\right] - \frac{G_i^2 - V_{oi}^2 \cos^2 \Phi_i}{2\, r_i^2}$$

where β_i^2 is a constant at our disposal, and $\sin \Phi_i = e_i \quad , \quad V_{oi}^2 = \beta_i^2 a_i$.

It is customary to take $\beta_i^2 = f m_o \, (1 + m_i)$. By leaving β_i still undetermined however we have a parameter at our disposal that later may be so determined that the intermediate orbit in the theory of longitudes assumes the desired form. **)

Using the relations:

$$F_i = V_{oi} \, (1 - \cos \Phi_i)$$

$$r_i = \frac{a_i \, (1 - e_i^2)}{1 + e_i \cos f_i}$$

it appears that the elements ψ_i and Ψ_i do not enter into the expression of T_i.

Further we have:

$$\Omega_i = f \, \frac{m_o + m_i}{r_i} + R_i \, .$$

The first part of the second side does not contain the variables ψ_i and Ψ_i. Consequently it is allowable to substitute in the equations (A) instead of H_i the perturbative function R_i with the negative sign.

The equations (A) then become:

$$\text{(B)} \qquad \frac{d\psi_i}{dt} = -\frac{\partial R_i}{\partial \Psi_i} \quad , \quad \frac{d\Psi_i}{dt} = \frac{\partial R_i}{\partial \psi_i} \, .$$

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*) Cf. Footnote p. 20, Leiden XII, 1.

**) Cf. Leiden XII, 1, p. 13.

We now introduce instead of the longitude of node ω_i a variable ϑ_i, defined by:

$$\vartheta_i = \omega_i - \sigma_i$$

where σ_i*) is determined by:

$$\frac{d\sigma_i}{dt} = -\frac{\partial R_i}{\partial G_i}$$

for which — since $\Psi_i = \frac{1}{2} G_i \gamma_i^2 \left(\gamma_i = 2 \sin \frac{1}{2} i_i\right)$ — we can also write:

$$\frac{d\sigma_i}{dt} = +\frac{1}{2} \frac{\gamma_i}{G_i} \frac{\partial R_i}{\partial \gamma_i}.$$

Now we have:

$$\frac{d\omega_i}{dt} = -\frac{d\psi_i}{dt} = \frac{\partial R_i}{\partial \Psi_i} = \frac{1}{\gamma_i G_i} \frac{\partial R_i}{\partial \gamma_i}$$

and thus:

$$\frac{d\sigma_i}{dt} = \frac{\gamma_i^2}{2} \frac{d\omega_i}{dt}$$

which shows that, with respect to ω_i, σ_i is a quantity of the second order and may be regarded as a small correction to ω_i.

We then find:

(C)
$$\frac{d\vartheta_i}{dt} = \cos i_i \frac{\partial R_i}{\partial \Psi_i}, \quad \frac{d\Psi_i}{dt} = \frac{\partial R_i}{\partial \psi_i}.$$

Finally we introduce our final variables P_i and Q_i, defined by:

(2)
$$\begin{cases} P_i = \Gamma_i \cos \vartheta_i \\ Q_i = -\Gamma_i \sin \vartheta_i, \end{cases}$$

where $\Gamma_i = \gamma_i \sqrt{G_i}$, and thus: $\Psi_i = \frac{\Gamma_i^2}{2}$.

These variables are nearly the same as Souillart's p_i and q_i. Only the longitude of the node has been reckoned by us in a direction opposite to that of Souillart, and instead of Souillart's ϕ_i we have taken $\gamma_i \sqrt{G_i}$. Because of the identity $QdP - \Psi d\vartheta = -\frac{1}{2} d(\Psi \sin 2\vartheta)$, the transformation (2) satisfies the criterion of canonicity, whence, if the equations (C) were a canonical set, we could immediately write down the new equations. This, however, is not the case. We therefore proceed in the following way:

By differentiating the equations (2), we get:

$$\frac{dP_i}{dt} = \frac{d\Gamma_i}{dt} \cos \vartheta_i - \Gamma_i \sin \vartheta_i \frac{d\vartheta_i}{dt}$$

$$\frac{dQ_i}{dt} = -\frac{d\Gamma_i}{dt} \sin \vartheta_i - \Gamma_i \cos \vartheta_i \frac{d\vartheta_i}{dt}$$

or also:

(D)
$$\frac{dP_i}{dt} = \frac{P_i}{\Gamma_i} \frac{d\Gamma_i}{dt} + Q_i \frac{d\vartheta_i}{dt}$$

$$\frac{dQ_i}{dt} = \frac{Q_i}{\Gamma_i} \frac{d\Gamma_i}{dt} - P_i \frac{d\vartheta_i}{dt},$$

*) Cf. Leiden XII, 1, p. 13.

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From $\Gamma_i^2 = 2\Psi_i$ follows:

$$\frac{d\Gamma_i}{dt} = \frac{1}{\Gamma_i}\frac{d\Psi_i}{dt} = -\frac{1}{\Gamma_i}\frac{\partial R_i}{\partial \vartheta_i} \tag{3}$$

we also have:

$$\frac{d\vartheta_i}{dt} = \frac{\cos i_i}{\gamma_i G_i}\frac{\partial R_i}{\partial \gamma_i} = \frac{1-\frac{1}{2}\gamma_i^2}{\gamma_i G_i}\frac{\partial R_i}{\partial \gamma_i}. \tag{4}$$

In order to express $\frac{d\Gamma_i}{dt}$ and $\frac{d\vartheta_i}{dt}$ in terms of P_i and Q_i, as is desirable, we employ the relations:

$$\begin{aligned} \frac{\partial R_i}{\partial \vartheta_i} &= Q_i\frac{\partial R_i}{\partial P_i} - P_i\frac{\partial R_i}{\partial Q_i} \\ \frac{\partial R_i}{\partial \gamma_i} &= \frac{\partial R_i}{\partial \Gamma_i}\sqrt{G_i} = \frac{P_i\sqrt{G_i}}{\Gamma_i}\frac{\partial R_i}{\partial P_i} + \frac{Q_i\sqrt{G_i}}{\Gamma_i}\frac{\partial R_i}{\partial Q_i}. \end{aligned} \tag{5}$$

Inserting these expressions in (3) and (4) and substituting the obtained values for $\frac{d\Gamma_i}{dt}$ and $\frac{d\vartheta_i}{dt}$ in the first equation (D) we get after some reductions:

$$\frac{dP_i}{dt} = \frac{\partial R_i}{\partial Q_i} - \frac{1}{2}\frac{Q_i}{G_i}\gamma_i\frac{\partial R_i}{\partial \gamma_i},$$

and also:

$$\frac{dQ_i}{dt} = -\frac{\partial R_i}{\partial P_i} + \frac{1}{2}\frac{P_i}{G_i}\gamma_i\frac{\partial R_i}{\partial \gamma_i}.$$

As might be expected these equations are not canonical. Their right hand side consists of two parts of different order of magnitude. The first terms are of the first degree in γ_i, the second part is of the third degree.

Instead of the equations (D) we thus have obtained the set:

$$\begin{aligned} \frac{dP_{io}}{dt} &= \frac{\partial R_i}{\partial Q_{io}} - \frac{1}{2}\frac{Q_{io}}{G_i}\gamma_{io}\frac{\partial R_i}{\partial \gamma_{io}}, \\ \frac{dQ_{io}}{dt} &= -\frac{\partial R_i}{\partial P_{io}} + \frac{1}{2}\frac{P_{io}}{G_i}\gamma_{io}\frac{\partial R_i}{\partial \gamma_{io}}. \end{aligned} \tag{D$_1$}$$

We have added the suffix o to indicate that the inclinations and nodes are referred to a fixed fundamental plane. Souillart throughout the whole of his work has preserved the same fixed plane of (x, y). But there are reasons which make it advantageous to transform our equations to a moving fundamental plane. The perturbative function employed by us is a rather complicated one, much more complicated than that of Souillart. In this function occur the latitudes s_i of the satellites referred to Jupiter's true equator. In order to keep the perturbative function in its present form and to avoid rendering it still more complicated, we prefer a transformation of our equations to Jupiter's equator as fundamental plane.

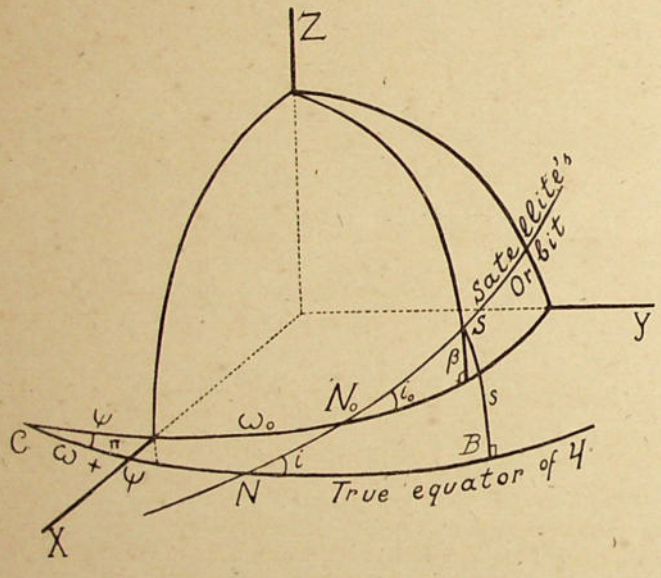
Transformation of the equations.

We will begin by giving particulars about the notations used. All the quantities which refer to the old fixed plane of (x,y) are indicated by the suffix o, all which relate to the new moving plane of (x,y) i. e. Jupiter's equator are without the suffix.

Further the longitudes will now be measured along the fixed plane of (x,y) up to the node of ♃'s true equator on this plane, and from there along this equator. The longitude of the node measured in this way is ω_i.

β_i will denote the latitude of a satellite with respect to the fixed plane of (x,y).

w_{io} is the true longitude of a satellite in its orbit and is measured along the fixed plane of (x,y) up to its node and further along the orbital plane of the satellite; w_i is the corresponding quantity referred to the moving plane of (x,y).



The longitude of the descending node of Jupiter's true equator on the fixed plane of (x,y) is $-\psi$. In order to transform the equations (D_t) we derive the relation between the variables

$$p_{io} = \sin i_{io} \cos \vartheta_{io}$$
$$q_{io} = -\sin i_{io} \sin \vartheta_{io}$$

referring to the fixed plane of (x,y), and:

$$p_i = \sin i_i \cos \vartheta_i$$
$$q_i = -\sin i_i \sin \vartheta_i$$

referring to Jupiter's equator.

Applying to the spherical triangle CNN_o the sine and sine-cosine formulae of spherical trigonometry we find:

$$\sin i_i \sin(\omega_i + \psi) = \sin i_{io} \sin(\omega_{io} + \psi)$$
$$\sin i_i \cos(\omega_i + \psi) = \cos(\omega_{io} + \psi) \sin i_{io} \cos\pi + \sin\pi \cos i_{io}. \quad (6)$$

As in these expressions instead of ω_i and ω_{io} we need the variables ϑ_i and ϑ_{io}, we introduce, in analogy to the substitutions:

$$\omega_{io} = \vartheta_{io} + \sigma_i \quad , \quad \omega_i = \vartheta_i + \sigma_i .$$

a new variable: $\psi_i' = \psi + \sigma_i$.

We then obtain instead of (6) the following set:

$$\begin{aligned} &\sin i_i \sin(\psi_i' + \vartheta_i) = \sin i_{io} \sin(\psi_i' + \vartheta_{io}) \\ &\sin i_i \cos(\psi_i' + \vartheta_i) = \cos(\psi_i' + \vartheta_{io}) \sin i_{io} \cos\pi + \sin\pi \cos i_{io} . \end{aligned} \tag{7}$$

Multiplying these equations by $\sin\psi_i'$ and $\cos\psi_i'$ respectively and adding them, we find after some reductions:

$$\sin i_i \cos\vartheta_i = \sin i_{io} \cos\vartheta_{io} + \sin\pi \cos\psi_i' \left[\cos i_{io} - \sin i_{io} \tan\frac{\pi}{2} \cos(\vartheta_{io} + \psi_i')\right]$$

and similarly:

$$-\sin i_i \sin\vartheta_i = -\sin i_{io} \sin\vartheta_{io} + \sin\pi \sin\psi_i' \left[\cos i_{io} - \sin i_{io} \tan\frac{\pi}{2} \cos(\vartheta_{io} + \psi_i')\right].$$

Putting in analogy to the former expressions of α and β:

$$\begin{aligned} \alpha_i' &= \sin\pi \sin\psi_i' \\ \beta_i' &= \sin\pi \cos\psi_i' \end{aligned}$$

we find:

$$\begin{aligned} p_i &= p_{io} + \beta_i' \left[\cos i_{io} - \sin i_{io} \tan\frac{\pi}{2} \cos(\vartheta_{io} + \psi_i')\right] \\ q_i &= q_{io} + \alpha_i' \left[\cos i_{io} - \sin i_{io} \tan\frac{\pi}{2} \cos(\vartheta_{io} + \psi_i')\right]. \end{aligned}$$

Now in our units to be stated presently i_{io} is smaller than 10^{-2} and π is of the order of 10^{-6}, whence, neglecting quantities of the order of 10^{-14}, which is quite sufficient for our purpose, we may put:

$$\begin{aligned} p_i &= p_{io} + \beta_i' \cos i_{io} \\ q_i &= q_{io} + \alpha_i' \cos i_{io} \end{aligned}$$

The β_i' and α_i', however, do not directly determine the position of the equator of $\mathfrak{L}$, because they include the quantity σ_i. But with the degree of accuracy, which P_i and Q_i are subjected to (they must be known in 6 decimal places, see Chapter III), the β_i' and α_i' may be replaced by α_i and β_i, since we have:

$$\begin{aligned} \beta_i' &= \sin\pi \cos\psi_i' = \beta - \alpha \sin\sigma_i - 2\beta \sin^2\frac{1}{2}\sigma_i \\ \alpha_i' &= \sin\pi \sin\psi_i' = \alpha + \beta \sin\sigma_i - 2\alpha \sin^2\frac{1}{2}\sigma_i . \end{aligned}$$

Now σ_i is determined by the equation:

$$\frac{d\sigma_i}{dt} = \frac{\gamma_i}{2G_i} \frac{\partial R_i}{\partial \gamma_i} .$$

As $\frac{\partial R_i}{\partial \gamma_i}$ is of the order of γ_i, σ_i is of the order γ_i^2, i. e. of the order 10^{-4} at the utmost. Thus neglecting quantities of an order 10^{-10}, we may put:

$$\beta_i' = \beta \quad , \quad \alpha_i' = \alpha$$

and consequently:

$$\begin{aligned} p_i &= p_{io} + \beta \cos i_{io} \\ q_i &= q_{io} + \alpha \cos i_{io} . \end{aligned} \tag{8}$$

By means of these equations the relations between P_i, Q_i and P_{io}, Q_{io} can easily be derived. We have:

$$P_{io} = \quad 2 \sin \tfrac{1}{2} i_{io} \sqrt{G_i} \cos \vartheta_{io}$$

$$Q_{io} = - 2 \sin \tfrac{1}{2} i_{io} \sqrt{G_i} \sin \vartheta_{io}$$

whence:

$$p_{io} = \frac{P_{io}}{\sqrt{G_i}} \cos \tfrac{1}{2} i_{io}$$

$$q_{io} = \frac{Q_{io}}{\sqrt{G_i}} \cos \tfrac{1}{2} i_{io} .$$

Substituting these in (8) we find:

$$p_i = \frac{P_{io}}{\sqrt{G_i}} \cos \tfrac{1}{2} i_{io} + \beta \cos i_{io}$$

$$q_i = \frac{Q_{io}}{\sqrt{G_i}} \cos \tfrac{1}{2} i_{io} + \alpha \cos i_{io}$$

and thus:

$$\begin{aligned} P_i &= P_{io} \cos \tfrac{1}{2} i_{io} + \beta \sqrt{G_i} \cos i_{io} \\ Qi &= Q_{io} \cos \tfrac{1}{2} i_{io} + \alpha \sqrt{G_i} \cos i_{io} . \end{aligned} \tag{9}$$

Now we have put:

$$\gamma_{io} = 2 \sin \tfrac{1}{2} i_{io}$$

and thus, neglecting quantities of an order higher than i_{io}^3, we have:

$$\cos \tfrac{1}{2} i_{io} = 1 - \tfrac{1}{8} \gamma_{io}^2$$

$$\cos i_{io} = 1 - \tfrac{1}{2} \gamma_{io}^2 .$$

It might be doubted if it is permissible to neglect i_{io}^3 in the case of the Sun also, for the inclination of the Sun is of the order 10^{-1}, but if we inspect the influence of the Sun on the motions of the satellites and Jupiter's equator more closely, it appears to be allowable to omit the terms with i_{io}^3 even when the index i refers to the Sun. Employing these developments in (9) we finally find the following transformation-formulae:

$$\begin{aligned} P_i &= P_{io} \left(1 - \tfrac{1}{8} \gamma_{io}^2\right) + \beta \sqrt{G_i} \left(1 - \tfrac{1}{2} \gamma_{io}^2\right) \\ Q_i &= Q_{io} \left(1 - \tfrac{1}{8} \gamma_{io}^2\right) + \alpha \sqrt{G_i} \left(1 - \tfrac{1}{2} \gamma_{io}^2\right) . \end{aligned} \tag{10}$$

Differentiating the first of the equations (10) with respect to t, we find:

$$(11)\qquad \frac{dP_i}{dt}=\left(1-\tfrac{1}{8}\gamma_{io}^{\,2}\right)\frac{dP_{io}}{dt}-\left(\tfrac{1}{4}P_{io}+\beta\sqrt{G_i}\right)\gamma_{io}\frac{d\gamma_{io}}{dt}+\left(1-\tfrac{1}{2}\gamma_{io}^{\,2}\right)\frac{\beta}{2\sqrt{G_i}}\frac{dG_i}{dt}+\left(1-\tfrac{1}{2}\gamma_{io}^{\,2}\right)\sqrt{G_i}\frac{d\beta}{dt}.$$

On the right hand side of (11) occur, because of $\frac{dP_{io}}{dt}$, terms containing $\frac{\partial R_i}{\partial Q_{io}}$ and $\gamma_i\,\frac{\partial R_i}{\partial \gamma_{io}}$. Now the reason why we have transformed the equations (D_1) is to leave the perturbative function R_i unaltered and develop it in terms of inclinations referring to Jupiter's equator (thus e. g. in terms of p_i and q_i).

The terms $\frac{\partial R_i}{\partial Q_{io}}$ and $\gamma_{io}\,\frac{\partial R_i}{\partial \gamma_{io}}$ must therefore be expressed in $\frac{\partial R_i}{\partial p_i}$ and $\frac{\partial R_i}{\partial q_i}$. In order to obtain these expressions we start from the relations:

$$(12)\qquad \begin{aligned} p_i &= p_{io}'\cos\tfrac{1}{2}i_{io}+\beta\cos i_{io}\\ q_i &= q_{io}'\cos\tfrac{1}{2}i_{io}+\alpha\cos i_{io}\end{aligned}$$

where we have put:

$$p_{io}'=\frac{P_{io}}{\sqrt{G_i}}\quad,\quad q_{io}'=\frac{Q_{io}}{\sqrt{G_i}}.$$

According to (12) p_i and q_i do not depend directly on P_{io} and Q_{io} but through the variables p_{io}' and q_{io}'. Thus we have:

$$(13)\qquad \begin{aligned}\frac{\partial R_i}{\partial Q_{io}}&=\frac{\partial R_i}{\partial q_i}\cdot\frac{\partial q_i}{\partial q_{io}'}\cdot\frac{\partial q_{io}'}{\partial Q_{io}}+\frac{\partial R_i}{\partial p_i}\cdot\frac{\partial p_i}{\partial q_{io}'}\cdot\frac{\partial q_{io}'}{\partial Q_{io}}\\ \frac{\partial R_i}{\partial P_{io}}&=\frac{\partial R_i}{\partial q_i}\cdot\frac{\partial q_i}{\partial p_{io}'}\cdot\frac{\partial p_{io}'}{\partial P_{io}}+\frac{\partial R_i}{\partial p_i}\cdot\frac{\partial p_i}{\partial p_{io}'}\cdot\frac{\partial p_{io}'}{\partial P_{io}}.\end{aligned}$$

Further we find:

$$\frac{\partial p_{io}'}{\partial P_{io}}=\frac{\partial q_{io}'}{\partial Q_{io}}=\frac{1}{\sqrt{G_i}}.$$

From the equations (12) we derive:

$$(14)\qquad \begin{aligned}\frac{\partial q_i}{\partial q_{io}'}&=\cos\tfrac{1}{2}i_{io}+q_{io}'\frac{\partial\cos\frac{1}{2}i_{io}}{\partial q_{io}'}+\alpha\frac{\partial\cos i_{io}}{\partial q_{io}'}\\ \frac{\partial p_i}{\partial q_{io}'}&=p_{io}'\frac{\partial\cos\frac{1}{2}i_{io}}{\partial q_{io}'}+\beta\frac{\partial\cos i_{io}}{\partial q_{io}'}.\end{aligned}$$

With sufficient approximation we have:

$$\begin{aligned}\frac{\partial\cos\frac{1}{2}i_{io}}{\partial q_{io}'}&=-\tfrac{1}{4}\gamma_{io}\frac{\partial\gamma_{io}}{\partial q_{io}'}=-\tfrac{1}{4}q_{io}'\\ \frac{\partial\cos i_{io}}{\partial q_{io}'}&=-\gamma_{io}\frac{\partial\gamma_{io}}{\partial q_{io}'}=-p_{io}'\end{aligned}$$

by using the relation:

$$p_{io}^{\,2}+q_{io}^{\,2}=\gamma_{io}^{\,2}.$$

Inserting these values in (14) we obtain at last:

$$\frac{\partial q_i}{\partial q_{io}'} = 1 - \tfrac{1}{8}\gamma_{io}^2 - \tfrac{1}{4}q_{io}'^2 - \alpha q_{io}'$$

$$\frac{\partial p_i}{\partial q_{io}'} = -\tfrac{1}{4}p_{io}'q_{io}' - \beta q_{io}'$$

and similarly:

$$\frac{\partial p_i}{\partial p_{io}'} = 1 - \tfrac{1}{8}\gamma_{io}^2 - \tfrac{1}{4}p_{io}'^2 - \beta p_{io}'$$

$$\frac{\partial q_i}{\partial p_{io}'} = -\tfrac{1}{4}p_{io}'q_{io}' - \alpha p_{io}'.$$

Substituting the above relations in (13) we finally find:

$$\frac{\partial R_i}{\partial Q_{io}} = \frac{1}{\sqrt{G_i}}\left[\frac{\partial R_i}{\partial q_i}\left(1 - \tfrac{1}{8}\gamma_{io}^2 - \tfrac{1}{4}q_{io}'^2\right) + \frac{\partial R_i}{\partial p_i}\left(-\tfrac{1}{4}p_{io}'q_{io}'\right)\right]$$

(15)

$$\frac{\partial R_i}{\partial P_{io}} = \frac{1}{\sqrt{G_i}}\left[\frac{\partial R_i}{\partial p_i}\left(1 - \tfrac{1}{8}\gamma_{io}^2 - \tfrac{1}{4}p_{io}'^2\right) + \frac{\partial R_i}{\partial q_i}\left(-\tfrac{1}{4}p_{io}'q_{io}'\right)\right]$$

where quantities of an order exceeding γ_{io}^3 have been omitted.

In the second place we have to derive the formulae for $\gamma_{io}\frac{\partial R_i}{\partial \gamma_{io}}$ in terms of $\frac{\partial R_i}{\partial p_i}$ and $\frac{\partial R_i}{\partial q_i}$.

We have, employing the relation $p_{io}^2 + q_{io}^2 = \gamma_{io}^2$:

$$\frac{\partial R_i}{\partial \gamma_{io}} = \frac{p_{io}}{\gamma_{io}}\frac{\partial R_i}{\partial p_{io}} + \frac{q_{io}}{\gamma_{io}}\frac{\partial R_i}{\partial q_{io}}$$

whence also:

(16)
$$\gamma_{io}\frac{\partial R_i}{\partial \gamma_{io}} = p_{io}\frac{\partial R_i}{\partial p_{io}} + q_{io}\frac{\partial R_i}{\partial q_{io}}$$

which, by means of (15) may immediately be expressed in terms of $\frac{\partial R_i}{\partial p_i}$ and $\frac{\partial R_i}{\partial q_i}$.

There remains one term in (11) which refers to the former fixed plane of (x, y) and is now to be transformed to Jupiter's equator, viz:

(17)
$$-\gamma_{io}\frac{d\gamma_{io}}{dt}\left(\tfrac{1}{4}P_{io} + \beta\sqrt{G_i}\right).$$

We have put:

$$\gamma_{io} = \frac{\Gamma_{io}}{\sqrt{G_i}}$$

and thus get:

(18)
$$\frac{d\gamma_{io}}{dt} = \frac{1}{\sqrt{G_i}}\frac{d\Gamma_{io}}{dt} - \frac{\Gamma_{io}}{2G_i\sqrt{G_i}}\frac{dG_i}{dt}.$$

Making use of the second equation (C), we find, since $\Gamma_i^2 = 2\Psi_i$:

$$\frac{d\gamma_{io}}{dt} = -\frac{1}{\Gamma_{io}\sqrt{G_i}}\frac{\partial R_i}{\partial \vartheta_{io}} - \frac{\gamma_{io}}{2G_i}\frac{dG_i}{dt}.$$

Hence we have:

$$\gamma_{io}\frac{d\gamma_{io}}{dt} = -\frac{1}{G_i}\frac{\partial R_i}{\partial \vartheta_{io}} - \frac{\gamma_{io}}{2G_i}\frac{dG_i}{dt}.$$

Since:

$$\frac{\partial R_i}{\partial \vartheta_{io}} = \frac{\partial R_i}{\partial p_{io}}q_{io} - \frac{\partial R_i}{\partial q_{io}}p_{io},$$

we find:

$$\gamma_{io}\frac{d\gamma_{io}}{dt} = -\frac{1}{G_i}\left(q_{io}\frac{\partial R_i}{\partial p_{io}} - p_{io}\frac{\partial R_i}{\partial q_{io}}\right) - \frac{\gamma_{io}^2}{2G_i}\frac{dG_i}{dt}.$$

This expression shows that $\gamma_{io}\frac{d\gamma_{io}}{dt}$ is of the second order in the inclinations. Our aim is to derive the differential equations for the inclinations and nodes, including the terms of the third order in the inclinations.

Accordingly we may put in (17)

$$P_{io} = P_i \quad , \quad Q_{io} = Q_i \quad , \quad p_{io} = p_i \quad , \quad q_{io} = q_i$$

which is easily verified by means of (15) and (12).

Then (17) assumes the form:

$$(19)\quad -\gamma_{io}\frac{d\gamma_{io}}{dt}\left(\tfrac{1}{4}P_{io} + \beta_i\sqrt{G_i}\right) = -\left(\tfrac{1}{4}P_i + \beta\sqrt{G_i}\right)\left[-\frac{1}{G_i}\left(q_i\frac{\partial R_i}{\partial p_i} - p_i\frac{\partial R_i}{\partial q_i}\right) - \frac{\gamma_i^2}{2G_i}\frac{dG_i}{dt}\right].$$

If we now substitute the equations (15), (16) and (19) in (11), we find by using the equations (D_4):

$$\frac{dP_i}{dt} = \left(1-\tfrac{1}{8}\gamma_{io}^2\right)\left[\frac{1}{\sqrt{G_i}}\left\{\frac{\partial R_i}{\partial q_i}\left(1-\tfrac{1}{8}\gamma_{io}^2-\tfrac{1}{4}q_{io}'^2\right) + \frac{\partial R_i}{\partial p_i}\left(-\tfrac{1}{4}p_{io}'q_{io}'\right)\right\}\right]$$

$$-\tfrac{1}{2}\left(1-\tfrac{1}{8}\gamma_{io}^2\right)\frac{Q_{io}}{G_i}\left(p_{io}\frac{\partial R_i}{\partial p_{io}} + q_{io}\frac{\partial R_i}{\partial q_{io}}\right) - \left(\tfrac{1}{4}P_i + \beta\sqrt{G_i}\right)\left[-\frac{1}{G_i}\left(\frac{\partial R_i}{\partial p_i}q_i - \frac{\partial R_i}{\partial q_i}p_i\right) - \frac{\gamma_i^2}{2G_i}\frac{dG_i}{dt}\right]$$

$$+\left(1-\tfrac{1}{2}\gamma_{io}^2\right)\frac{\beta}{2\sqrt{G_i}}\frac{dG_i}{dt} + \left(1-\tfrac{1}{2}\gamma_{io}^2\right)\sqrt{G_i}\frac{d\beta}{dt}.$$

Omitting terms of a degree higher than γ_i^3, we may in this expression, instead of p_{io}', q_{io}', γ_{io}, take p_i, q_i, γ_i respectively. Substituting also $\gamma_i^2 = p_i^2 + q_i^2$ and gathering corresponding terms, we finally find:

$$(E)\quad \frac{dP_i}{dt} = \frac{1}{\sqrt{G_i}}\left[\left(1-q_i^2-\tfrac{1}{2}p_i^2\right)\frac{\partial R_i}{\partial q_i} - \tfrac{1}{2}p_iq_i\frac{\partial R_i}{\partial p_i}\right] + \tfrac{1}{2}\left\{\beta + \tfrac{1}{4}p_i^3 + \tfrac{1}{4}p_iq_i^2\right\}\frac{1}{\sqrt{G_i}}\frac{dG_i}{dt}$$

$$+\left(1-\tfrac{1}{2}p_i^2-\tfrac{1}{2}q_i^2\right)\sqrt{G_i}\frac{d\beta}{dt}$$

and similarly:

$$(E)\quad \frac{dQ_i}{dt} = -\frac{1}{\sqrt{G_i}}\left[\left(1-p_i^2-\tfrac{1}{2}q_i^2\right)\frac{\partial R_i}{\partial p_i} - \tfrac{1}{2}p_iq_i\frac{\partial R_i}{\partial q_i}\right] + \tfrac{1}{2}\left\{\alpha + \tfrac{1}{4}p_i^2q_i + \tfrac{1}{4}q_i^3\right\}\frac{1}{\sqrt{G_i}}\frac{dG_i}{dt}$$

$$+\left(1-\tfrac{1}{2}p_i^2-\tfrac{1}{2}q_i^2\right)\sqrt{G_i}\frac{d\alpha}{dt}.$$

The differential coefficients $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$ occurring in (E) determine the precession and nutation of Jupiter's equator. In order to express them in terms of P_i and Q_i, we start from the expressions:

$$\begin{aligned} \alpha &= \sin\psi \sin\pi \\ \beta &= \cos\psi \sin\pi . \end{aligned}$$

By differentiating these expressions we find:

$$(20) \qquad \begin{aligned} \frac{d\alpha}{dt} &= \cos\psi \sin\pi \frac{d\psi}{dt} + \sin\psi \cos\pi \frac{d\pi}{dt}, \\ \frac{d\beta}{dt} &= -\sin\psi \sin\pi \frac{d\psi}{dt} + \cos\psi \cos\pi \frac{d\pi}{dt}. \end{aligned}$$

From the theory of precession and nutation we take the following well known formulae: *)

$$\begin{aligned} \frac{d\psi}{dt} &= -\frac{1}{Cn \sin\pi} \frac{\partial U}{\partial \pi} \\ \frac{d\pi}{dt} &= +\frac{1}{Cn \sin\pi} \frac{\partial U}{\partial \psi} \end{aligned}$$

where C is the principal moment of inertia of Jupiter with reference to its axis of revolution, n the velocity of rotation around that axis and the force-function U is the same as our potential function, thus:

$$U = \Sigma U_i$$

where the sum is to be taken over the four satellites and the Sun. For the same reasons as formerly stated the influence of Saturn and the other planets may be neglected.

Further we have:

$$\begin{aligned} \frac{\partial U}{\partial \pi} &= \frac{\partial U}{\partial \alpha} \sin\psi \cos\pi + \frac{\partial U}{\partial \beta} \cos\psi \cos\pi \\ \frac{\partial U}{\partial \psi} &= \frac{\partial U}{\partial \alpha} \cos\psi \sin\pi - \frac{\partial U}{\partial \beta} \sin\psi \sin\pi . \end{aligned}$$

Substituting the above relations in (20) we find, after some reductions:

$$(21) \qquad \begin{aligned} \frac{d\alpha}{dt} &= -\frac{\cos\pi}{Cn} \frac{\partial U}{\partial \beta} \\ \frac{d\beta}{dt} &= \frac{\cos\pi}{Cn} \frac{\partial U}{\partial \alpha} \end{aligned}$$

where:

$$U = f m_0 \Sigma_i \left[\frac{1}{r_i} + \frac{Jb^2}{3 r_i^3} \left(1 - 3 \sin^2 s_i\right) + \frac{Kb^4}{10 r_i^5} \left(1 - 10 \sin^2 s_i + \frac{35}{3} \sin^4 s_i\right) \right] m_i$$

and: $\qquad i = 1, \ldots . 4, \odot$

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*) Cf. Tisserand II p. 427.

It appears that U contains the variables p_i and q_i, and thus:

$$\frac{\partial U}{\partial \beta} = \Sigma \frac{\partial U_i}{\partial p_i} \cdot \frac{\partial p_i}{\partial \beta} = \Sigma \cos i_{io} \frac{\partial U_i}{\partial p_i}$$

(22)

$$\frac{\partial U}{\partial \alpha} = \Sigma \frac{\partial U_i}{\partial q_i} \cdot \frac{\partial q_i}{\partial \alpha} = \Sigma \cos i_{io} \frac{\partial U_i}{\partial q_i}$$

because of the relations:

$$p_i = p_{io} + \beta \cos i_{io}$$
$$q_i = q_{io} + \alpha \cos i_{io}.$$

[In reality:

$$\frac{\partial p_i}{\partial \beta} = \cos i_{io} \cos \sigma_i = \frac{\partial q_i}{\partial \alpha}$$

$$\frac{\partial p_i}{\partial \alpha} = -\cos i_{io} \sin \sigma_i = -\frac{\partial q_i}{\partial \beta}$$

as is easily derived from:

$$p_i = p_{io} + \left(\beta - \alpha \sin \sigma_i - 2\beta \sin^2 \tfrac{1}{2}\sigma_i\right) \cos i_{io}$$

$$q_i = q_{io} + \left(\alpha + \beta \sin \sigma_i - 2\alpha \sin^2 \tfrac{1}{2}\sigma_i\right) \cos i_{io}.$$

Now:

$$\frac{d\sigma_i}{dt} = \frac{\gamma_i^2}{2} \frac{d\omega_i}{dt}$$

and with sufficient approxiniation we may regard $\frac{d\omega_i}{dt}$ as a periodic function. On integrating we thus find for σ_i a quantity of the second order in γ_i, multiplied by a periodic factor. Seeing that $\frac{\partial U_i}{\partial p_i}$ and $\frac{\partial U_i}{\partial q_i}$ are at least of the first order in p_i and q_i and moreover contain the small factors J_i, K_i, m_i it is allowable to put:

$$\frac{\partial p_i}{\partial \beta} = \cos i_{io} \quad , \quad \frac{\partial p_i}{\partial \alpha} = 0 \quad , \quad \frac{\partial q_i}{\partial \beta} = 0 \quad , \quad \frac{\partial q_i}{\partial \alpha} = \cos i_{io}$$

and these expressions agree with the values above employed.]

Omitting terms of the fourth order in the inclinations, U_i reduces to:

$$U_i = f m_o m_i \left[-\frac{Jb^2}{r_i^3} - \frac{Kb^4}{r_i^5}\right] \sin^2 s_i.$$

Accordingly the equations (21) become:

$$\frac{d\alpha}{dt} = -\Sigma_i \frac{\cos \pi \cos i_{io}}{Cn} \frac{\partial U_i}{\partial p_i} = -\frac{\cos \pi}{Cn} \Sigma \left\{1 - \tfrac{1}{2}\left(p_i^2 + q_i^2\right)\right\} \frac{\partial U_i}{\partial p_i}$$

$$\frac{d\beta}{dt} = \Sigma_i \frac{\cos \pi \cos i_{io}}{Cn} \frac{\partial U_i}{\partial q_i} = \frac{\cos \pi}{Cn} \Sigma \left\{1 - \tfrac{1}{2}\left(p_i^2 + q_i^2\right)\right\} \frac{\partial U_i}{\partial q_i}.$$

In the terms of the third degree we have taken $p_{io} = p_i$, $q_{io} = q_i$, and we have neglected terms of the fifth degree.

Inserting these values in the equations (E) we get:

$$\frac{dP_i}{dt} = \frac{1}{\sqrt{G_i}}\left[\left(1 - q_i^2 - \tfrac{1}{2}p_i^2\right)\frac{\partial R_i}{\partial q_i} - \tfrac{1}{2}p_i q_i \frac{\partial R_i}{\partial p_i}\right] + \tfrac{1}{2}\left\{\beta + \tfrac{1}{4}p_i^3 + \tfrac{1}{4}p_i q_i^2\right\}\frac{1}{\sqrt{G_i}}\frac{dG_i}{dt}$$
$$+ \sqrt{G_i}\left\{1 - \tfrac{1}{2}(p_i^2 + q_i^2)\right\}\frac{\cos\pi}{Cn}\sum_i\left\{1 - \tfrac{1}{2}(p_i^2 + q_i^2)\right\}\frac{\partial U_i}{\partial q_i}$$

and similarly $\frac{dQ_i}{dt}$.

With sufficient approximation we at last get the final equations:

(F)
$$\begin{cases} \dfrac{dP_i}{dt} = \dfrac{1}{\sqrt{G_i}}\left[\left(1 - q_i^2 - \tfrac{1}{2}p_i^2\right)\dfrac{\partial R_i}{\partial p_i} - \tfrac{1}{2}p_i q_i \dfrac{\partial R_i}{\partial p_i}\right] + \tfrac{1}{2}\left\{\beta + \tfrac{1}{4}p_i^3 + \tfrac{1}{4}p_i q_i^2\right\}\dfrac{1}{\sqrt{G_i}}\dfrac{dG_i}{dt} \\ \qquad + \sqrt{G_i}\,\dfrac{1 - \frac{1}{2}(\alpha^2 + \beta^2)}{Cn}\,\Sigma_j\left\{1 - \tfrac{1}{2}(p_i^2 + q_i^2 + p_j^2 + q_j^2)\right\}\dfrac{\partial U_j}{\partial q_j} \\ \dfrac{dQ_i}{dt} = -\dfrac{1}{\sqrt{G_i}}\left[\left(1 - p_i^2 - \tfrac{1}{2}q_i^2\right)\dfrac{\partial R_i}{\partial p_i} - \tfrac{1}{2}p_i q_i \dfrac{\partial R_i}{\partial q_i}\right] + \tfrac{1}{2}\left\{\alpha + \tfrac{1}{4}q_i^3 + \tfrac{1}{4}p_i^2 q_i\right\}\dfrac{1}{\sqrt{G_i}}\dfrac{dG_i}{dt} \\ \qquad - \sqrt{G_i}\,\dfrac{1 - \frac{1}{2}(\alpha^2 + \beta^2)}{Cn}\,\Sigma_j\left\{1 - \tfrac{1}{2}(p_i^2 + q_i^2 + p_j^2 + q_j^2)\right\}\dfrac{\partial U_j}{\partial p_j} \end{cases}$$

where $j = 1, \ldots 4, \odot$ and *the sums include the value* $j = i$.

On the right hand side of the equations (F) $\frac{dG_i}{dt}$ occurs. This quantity has to be derived from a separate theory of the longitudes. However, the effect of $\frac{dG_i}{dt}$ will be a very small one, as it is multiplied by a factor of the third order in the inclinations and $\frac{dG_i}{dt}$ will consist only of periodic parts. With our approximation we may therefore undoubtedly omit that part of the right hand side of (F).

Before closing this chapter we will recapitulate our final equations of motion as they will be used throughout the following, viz:

(F_1)
$$\begin{cases} \dfrac{dP_i}{dt} = \dfrac{1}{\sqrt{G_i}}\left[\left(1 - q_i^2 - \tfrac{1}{2}p_i^2\right)\dfrac{\partial R_i}{\partial q_i} - \tfrac{1}{2}p_i q_i \dfrac{\partial R_i}{\partial p_i}\right] + \dfrac{2 - \alpha^2 - \beta^2}{2Cn}\sqrt{G_i}\,\Sigma_j\left\{1 - \tfrac{1}{2}(p_i^2 + q_i^2 + p_j^2 + q_j^2)\right\}\dfrac{\partial U_j}{\partial q_j} \\ \dfrac{dQ_i}{dt} = -\dfrac{1}{\sqrt{G_i}}\left[\left(1 - p_i^2 - \tfrac{1}{2}q_i^2\right)\dfrac{\partial R_i}{\partial p_i} - \tfrac{1}{2}p_i q_i \dfrac{\partial R_i}{\partial q_i}\right] - \dfrac{2 - \alpha^2 - \beta^2}{2Cn}\sqrt{G_i}\,\Sigma_j\left\{1 - \tfrac{1}{2}(p_i^2 + q_i^2 + p_j^2 + q_j^2)\right\}\dfrac{\partial U_j}{\partial p_j} \\ \text{where } i = 1, \ldots 4;\ j = 1, \ldots 4, \odot, \text{ including } j = i. \end{cases}$$

(F_2)
$$\begin{cases} \dfrac{d\alpha}{dt} = -\dfrac{2 - \alpha^2 - \beta^2}{2\,Cn}\,\Sigma_i\left[1 - \tfrac{1}{2}(p_i^2 + q_i^2)\right]\dfrac{\partial U_i}{\partial p_i} \\ \dfrac{d\beta}{dt} = \dfrac{2 - \alpha^2 - \beta^2}{2\,Cn}\,\Sigma_i\left[1 - \tfrac{1}{2}(p_i^2 + q_i^2)\right]\dfrac{\partial U_i}{\partial q_i} \\ \text{where } i = 1, \ldots 4, \odot \end{cases}$$

(F_3) $\dfrac{d\sigma_i}{dt} = -\dfrac{1}{2G_i}\left(p_{io}\dfrac{\partial R_i}{\partial p_{io}} + q_{io}\dfrac{\partial R_i}{\partial q_{io}}\right)$, and $i = 1, \ldots 4$.

CHAPTER II.

DEVELOPMENT OF THE PERTURBATIVE FUNCTION IN TERMS OF THE INCLINATIONS.

The perturbative function is: (see pages 7 and 9)

$$R_i = f\,(m_\circ + m_i)\left\{\frac{1}{3}\frac{Jb^2}{r_i^3}\,(1 - 3\,\sin^2 s_i) + \frac{1}{10}\frac{Kb^4}{r_i^5}\,(1 - 10\,\sin^2 s_i + \frac{35}{3}\,\sin^4 s_i)\right\}$$

$$+ f\,m_\circ \Sigma\, m_j \left\{\frac{1}{\Delta_{ij}} - \frac{r_i}{r_j^2}\cos V_{ij}\left[1 + \frac{Jb^2}{r_j^2}\,(1 - 5\,\sin^2 s_j) + \frac{Kb^4}{r_j^4}\left(\frac{1}{2} - 7\,\sin^2 s_j + \frac{21}{2}\,\sin^4 s_j\right)\right]\right.$$

$$\left. - 2\,\frac{r_i}{r_j^2}\,\sin s_i\left[\frac{Jb^2}{r_j^2}\,\sin s_j + \frac{Kb^4}{r_j^4}\,(\sin s_j - \frac{7}{3}\,\sin^3 s_j)\right]\right\}$$

where we have put:

$$\frac{x_i x_j + y_i y_j + z_i z_j}{r_j^3} = \frac{r_i}{r_j^2}\cos V_{ij},$$

V_{ij} being the angle between the radius-vectors of the satellites i and j, whereas we also have: $\alpha x_i + \beta y_i + \gamma z_i = r_i \sin s_i$.

Further we shall use the following abbreviations:

$$J_i = \frac{Jb^2}{a_i^2}\quad,\quad K_i = \frac{Kb^4}{a_i^4}\quad,\quad \mathrm{r}_i = \frac{r_i}{a_i}\quad,\quad f\,(m_\circ + m_i) = 1 + \frac{m_i}{m_\circ} = \mu_i,$$

where we have chosen our units so as to make $f = 1$, $m_\circ = 1$.

The parts of the perturbative function which contain inclinations and nodes, neglecting in accordance with our limit of approximation terms higher than the fourth order in the inclinations, are as follows:

$$\mathrm{I}: -\frac{J_i\mu_i}{a_i \mathrm{r}_i^3}\sin^2 s_i \qquad \mathrm{IV}: -\frac{a_i}{a_j^2}\,m_j\left(\frac{J_j}{\mathrm{r}_j^3} + \frac{1}{2}\frac{K_j}{\mathrm{r}_j^4}\right)\frac{\mathrm{r}_i}{\mathrm{r}_j^2}\cos V_{ij}$$

$$\mathrm{II}: -\frac{K_i\mu_i}{a_i \mathrm{r}_i^5}\sin^2 s_i \qquad \mathrm{V}: -\frac{a_i}{a_j^2}\,m_j\left(5\,\frac{J_j}{\mathrm{r}_j^2} + 7\,\frac{K_j}{\mathrm{r}_j^4}\right)\frac{\mathrm{r}_i}{\mathrm{r}_j^2}\cos V_{ij}\sin^2 s_i$$

$$\mathrm{III}: \frac{m_j}{\Delta_{ij}} - \frac{a_i}{a_j^2}\,m_j\,\frac{\mathrm{r}_i}{\mathrm{r}_j^2}\cos V_{ij} \qquad \mathrm{VI}: -2\,\frac{a_i}{a_j^2}\,m_j J_j\,\frac{\mathrm{r}_i}{\mathrm{r}_j^4}\sin s_i \sin s_j$$

$$\mathrm{VII}: -2\,\frac{a_i}{a_j^2}\,m_j K_j\,\frac{\mathrm{r}_i}{\mathrm{r}_j^6}\sin s_i \sin s_j.$$

In the course of the development of these separate parts, we shall often omit terms on account of the smallness of the factors J_i, K_i, the eccentricities and masses of the satellites. For the proper understanding of the following it is therefore necessary to make some statements about the order of magnitude of these constants.

In accordance with the theory of the longitudes given by prof. de Sitter, we consider $\varkappa$, (which will be defined in Chapter III), as a quantity of the first order, Its value is about $\frac{1}{69}$. The masses are then of the second order (the largest is about $0{\cdot}8 \times 10^{-4}$), the inclinations and eccentricities are of the first order (the largest, $e_2 = 0{\cdot}009$), J_i is of the second and K_i is of the fourth order.

In the case of the Sun these statements are not exact, but the terms depending on the Sun do not cause any difficulty because of the large value of a_s, the mean distance of the Sun, and it is easy to verify that the developments given below for the satellites are also quite sufficient in the case of the Sun.

In the developments of the secular terms we shall go up to 10^{-12}, or which is the same, to γ^6, when not multiplied by a small factor. Thus we neglect terms like γ^7, $m\gamma^5$, $J\gamma^5$, $K\gamma^3$, $mJ\gamma^3$, $\gamma^3\sigma^2$, etc., σ being of the second order.

Moreover, in each special case, whe shall mention the terms omitted and the reason for the omission.

Development of I.

For the sake of simplicity we omit the factor $-\frac{J_i\mu_i}{a_i}$, remembering that it is of the second order.

We now have

$$\gamma_i = \sin i_i,$$

and therefore:

$$\sin s_i = \sin i_i \sin(w_i - \omega_i) = \gamma_i \sin(v_i - \vartheta_i),$$

from which it follows, that:

$$\sin^2 s_i = \gamma_i^2 \sin^2(v_i - \vartheta_i).$$

In order to get a development of r_i^{-3} in terms of mean anomalies, we can use well-known formulae for r_i^{-m} in terms of Besselian functions. But the basis of all the developments given here is *Hill's method*, which gives a development of the radius-vector in true anomalies, as published in the paper already quoted, but more in detail in his *Collected Works*, IV p. 400, from which we take, (f denoting the true anomaly):

$$r_i^{-3} = 1 + \frac{9}{2} e_i^2 + 3 e_i \cos f_i + \frac{3}{2} e_i^2 \cos 2 f_i + \ldots\ldots$$

In this expression we can introduce the mean anomaly l by the formula:

$$\cos f_i = \cos l_i - e_i + e_i \cos 2 l_i - \frac{9}{8} e_i^2 \cos l_i + \frac{9}{8} e_i^2 \cos 3 l_i + \ldots\ldots$$

To obtain the required accuracy we need not go beyond the first power of the eccentricities, but in the case of the secular terms, I have not omitted parts containing e_i^2. We then find:

(*a*) $$r_i^{-3} = 1 + \frac{3}{2} e_i^2 + 3 e_i \cos l_i + \frac{9}{2} e_i^2 \cos 2 l_i.$$

Now we have:

$$\gamma_i \sin(v_i - \vartheta_i) = p_i \sin v_i + q_i \cos v_i$$

whence:

$$(b) \qquad \gamma_i^2 \sin^2(v_i - \vartheta_i) = \tfrac{1}{2}(p_i^2 + q_i^2) + \tfrac{1}{2}(q_i^2 - p_i^2)\cos 2v_i + p_i q_i \sin 2v_i .$$

From the expressions (a) and (b) the development of $r_i^{-3} \sin^2 s_i$ in terms of the inclinations can be derived directly. We prefer, however, to introduce the mean anomalies instead of the true orbit-longitude v_i. This can be done by means of the relations:

$$w_i = f_i + \varpi_i$$
$$m_i = l_i + \varpi_i, \text{ where: } \varpi_i = \text{longitude of perijove}, \; m_i = \text{mean longitude}.$$

The expression (b), however, does not contain w_i, but $v_i = w_i - \sigma_i$. Now it has already been shown that σ_i is of the second order in the inclinations. Moreover I is multiplied by the factor f_i, which is of the second order in γ_i, and it is easy to verify, that, neglecting γ_i^6, we may identify w_i and v_i, by which the development is much simplified.

We now have:

$$w_i - m_i = v_i - m_i = f_i - l_i .$$

From the well known expression for the equation of the centre, we find:

$$(23) \qquad v_i - m_i = f_i - l_i = 2 e_i \sin l_i + \tfrac{5}{4} e_i^2 \sin 2 l_i .$$

It is thus clear, that $f_i - l_i$ or, which is the same, $v_i - m_i$ is a quantity of the first order.

Squaring (23) we find:

$$(v_i - m_i)^2 = 2 e_i^2 (1 - \cos 2 l_i),$$

where powers of e_i higher than e_i^3 have been neglected.

Further we have:

$$(24) \qquad \begin{aligned} \sin v_i &= \left[1 - \tfrac{1}{2}(v_i - m_i)^2\right] \sin m_i + (v_i - m_i)\cos m_i , \\ \cos v_i &= \left[1 - \tfrac{1}{2}(v_i - m_i)^2\right] \cos m_i - (v_i - m_i)\sin m_i , \\ \cos 2 v_i &= \left[1 - 2(v_i - m_i)^2\right] \cos 2 m_i - 2(v_i - m_i)\sin 2 m_i , \\ \sin 2 v_i &= \left[1 - 2(v_i - m_i)^2\right] \sin 2 m_i + 2(v_i - m_i)\cos 2 m_i , \end{aligned}$$

where we have put instead of $f_i - l_i$ its equivalent $v_i - m_i$ and $\cos(v_i - m_i)$ and $\sin(v_i - m_i)$ have been developed, neglecting quantities of the third order.

Substituting the relations (24) in (b) we finally find:

$$\begin{aligned} r_i^{-3}\gamma_i^2 \sin^2(v_i - \vartheta_i) = {} & \tfrac{1}{2}\left(1 + \tfrac{3}{2}e_i^2 + 3 e_i \cos l_i + \tfrac{9}{2} e_i^2 \cos 2 l_i\right)(p_i^2 + q_i^2) \\ & + \tfrac{1}{2}\left(1 + \tfrac{3}{2}e_i^2 + 3 e_i \cos l_i + \tfrac{9}{2} e_i^2 \cos 2 l_i\right)(q_i^2 - p_i^2)\left[\cos 2 m_i \left\{1 - 2(v_i - m_i)^2\right\} - 2 \sin 2 m_i (v_i - m_i)\right] \\ & + \left(1 + \tfrac{3}{2}e_i^2 + 3 e_i \cos l_i + \tfrac{9}{2} e_i^2 \cos 2 l_i\right) p_i q_i \left[\sin 2 m_i \left\{1 - 2(v_i - m_i)^2\right\} + 2 \cos 2 m_i (v_i - m_i)\right]. \end{aligned}$$

We shall now in general denote the secular part of a quantity X by $S[X]$ and the periodical part by $P[X]$.

We then find, using (23):

$$S\left[r_i^{-3}\gamma_i^2\sin^2(v_i-\vartheta_i)\right]=\tfrac{1}{2}(1+\tfrac{3}{2}e_i^2)(p_i^2+q_i^2)$$

$$P\left[r_i^{-3}\gamma_i^2\sin^2(v_i-\vartheta_i)\right]=\tfrac{1}{2}(q_i^2-p_i^2)\left[\cos 2(l_i+\varpi_i)-\tfrac{1}{2}e_i\cos(l_i+2\varpi_i)+3\tfrac{1}{2}e_i\cos(3l_i+2\varpi_i)\right]$$
$$+\quad p_iq_i\left[\sin 2(l_i+\varpi_i)-\tfrac{1}{2}e_i\sin(l_i+2\varpi_i)+3\tfrac{1}{2}e_i\sin(3l_i+2\varpi_i)\right]$$
$$+\tfrac{3}{2}(p_i^2+q_i^2)e_i\cos l_i.$$

Development of II.

The term II contains the quantity K_i which is of the fourth order; hence in this term also it is permissible to interchange w_i and v_i.

We have:

$$r_i^{-5}=1+10e_i^2+5e_i\cos f_i+5e_i^2\cos 2f_i+\ldots.$$

whence, substituting the value of $\cos f_i$ in terms of the mean anomaly:

$$r_i^{-5}=1+5e_i^2+5e_i\cos l_i+10e_i^2\cos 2l_i.$$

We thus find:

$$S\left[r_i^{-5}\gamma_i^2\sin^2(v_i-\vartheta_i)\right]=\tfrac{1}{2}(p_i^2+q_i^2)(1+5e_i^2)$$

$$P\left[r_i^{-5}\gamma_i^2\sin^2(v_i-\vartheta_i)\right]=\tfrac{1}{2}(q_i^2-p_i^2)\left[\cos(2l_i+2\varpi_i)+3e_i\cos(l_i+2\varpi_i)+7e_i\cos(3l_i+2\varpi_i)\right]$$
$$+\quad p_iq_i\left[\sin(2l_i+2\varpi_i)+3e_i\sin(l_i+2\varpi_i)+7e_i\sin(3l_i+2\varpi_i)\right]$$
$$+\tfrac{5}{2}(p_i^2+q_i^2)e_i\cos l_i.$$

Development of VI.

The chief part is, omitting the constant factors: $\frac{r_i}{r_j^4}\sin s_i\sin s_j$.

From the developments of the radius-vector in terms of the true anomaly, we take:

$$\frac{r_i}{r_j^4}=1+\tfrac{1}{2}e_i^2+3e_j^2+4e_j\cos f_j+7e_j^2\cos 2f_j-e_i\cos f_i-2e_ie_j\cos(f_i+f_j)-\tfrac{1}{2}e_i^2\cos 2f_i.$$

Making use of the expressions for $\sin s_i$ and $\sin s_j$, it is easy to verify that:

$$\sin s_i\sin s_j=\tfrac{1}{2}p_ip_j\left[\cos(v_i-v_j)-\cos(v_i+v_j)\right]+\tfrac{1}{2}q_iq_j\left[\cos(v_i+v_j)+\cos(v_i-v_j)\right]$$
$$+\tfrac{1}{2}p_iq_j\left[\sin(v_i+v_j)-\sin(v_j-v_i)\right]+\tfrac{1}{2}p_jq_i\left[\sin(v_i+v_j)+\sin(v_j-v_i)\right].$$

From these expressions it appears at once, that the development of VI in terms of the mean anomalies will be rather intricate. But we do not need an exact development, because of the small factors m_j and f_j. We may in fact entirely neglect products such as $m_j f_j p_j p_i e_i$. Therefore it is allowable to put the eccentricities equal to zero; we then have $r_ir_j^{-4}=1$, and instead of true longitudes v_i, v_j we can take mean longitudes m_i and m_j.

Hence we finally find:

$$r_i r_j^{-4} \sin s_i \sin s_j = \tfrac{1}{2}(p_i p_j + q_i q_j)\cos(m_i - m_j) + \tfrac{1}{2}(q_i q_j - p_i p_j)\cos(m_i + m_j)$$
$$+ \tfrac{1}{2}(p_i q_j + p_j q_i)\sin(m_i + m_j) + \tfrac{1}{2}(p_i q_j - p_j q_i)\sin(m_i - m_j)$$

in which the mean anomalies can be introduced by means of:

$$m_i - m_j = l_i - l_j + \varpi_i - \varpi_j$$
$$m_i + m_j = l_i + l_j + \varpi_i + \varpi_j .$$

It will be seen that VI contains only periodic terms multiplied by the very small factor $m_j J_j$ and it can be omitted, as having no importance for the further theory.

Development of VII.

The same approximation as in VI being allowed, it is easy to verify that VII does not contain either secular or critical, but only periodic terms. Hence, in consequence of the small factors occurring in VII, this term can also be neglected.

Development of IV.

Owing to the factors m_j, J_j, K_j that occur in this term, it is permissible to neglect the eccentricities in the developments of the radius-vector. In $\cos V_{ij}$ the terms containing inclinations are periodic, and at least of the second order in the inclinations, whence term IV can be neglected.

Development of V.

In this part of the perturbative function not only the eccentricities but also the inclinations in $\cos V_{ij}$ can be neglected. If we now put:

$$-\frac{a_j}{a_j^2}\, m_j\,(5J_j + 7K_j) = \mathfrak{A}_j$$

the development is:

$$\mathfrak{A}_j \cos(v_i - v_j)\sin^2 s_j = \tfrac{1}{2}\mathfrak{A}_j(p_j^2 + q_j^2)\cos(m_i - m_j) + \tfrac{1}{4}\mathfrak{A}_j(q_j^2 - p_j^2)\{\cos(m_i + m_j) + \cos(m_i - 3m_j)\}$$
$$-\tfrac{1}{2}\mathfrak{A}_j p_j q_j \{\sin(m_i + m_j) - \sin(m_i - 3m_j)\}$$

which only contains periodic terms.

From the above we conclude that only I and II contribute to the secular part of the perturbative function, and that the other parts can be neglected, except part III, which contains the reciprocal of the distance between two planets, that must now be developed.

Development of III.

The term III consists of two parts, viz: $\frac{m_j}{\Delta_{ij}}$ and $-\frac{a_i}{a_j^2}\, m_j \frac{r_i}{r_j^2}\cos V_{ij}$. The development of the latter is the simpler of the two parts and we begin with it.

A) Because of the small factor m_j and owing to the fact that $\cos V_{ij}$ is at least of the second order in the inclinations, we may neglect eccentricities and interchange true and mean longitudes. Also we can take $r_i = r_j = 1$.

In order to develop $\cos V_{ij}$ we use *Hill*'s formula *) and have:

$$\cos V_{ij} = (1 - \tfrac{1}{4} q_i^2)(1 - \tfrac{1}{4} q_j^2) \cos(v_i - v_j - \sigma_j + \sigma_i) + \tfrac{1}{4}(1 - \tfrac{1}{4} q_i^2) q_j^2 \cos(v_i + v_j - 2\vartheta_j + \sigma_i - \sigma_j)$$
$$+ \tfrac{1}{4}(1 - \tfrac{1}{4} q_j^2) q_i^2 \cos(v_i + v_j - 2\vartheta_i - \sigma_i + \sigma_j) + \tfrac{1}{2}(1 - \tfrac{1}{8} q_i^2 - \tfrac{1}{8} q_j^2) q_i q_j \cos(v_i - v_j - \vartheta_i + \vartheta_j)$$
$$- \tfrac{1}{2}(1 - \tfrac{1}{8} q_i^2 - \tfrac{1}{8} q_j^2) q_i q_j \cos(v_i + v_j - \vartheta_i - \vartheta_j) + \tfrac{1}{16} q_i^2 q_j^2 \cos(v_i - v_j - 2\vartheta_i + 2\vartheta_j - \sigma_i + \sigma_j)$$

where

$$q_i = 2 \sin \tfrac{1}{2} i_i = \gamma_i + \tfrac{1}{8} \gamma_i^3 .$$

If now in the expression of $\cos V_{ij}$ in the terms containing inclinations we neglect σ_i and σ_j, which amounts to neglecting at least $m_j \sigma_j q_j^2$, we may put, in accordance with *Hill*:

$$\cos V_{ij} = \cos V_{ij}^{\circ} + \delta_{ij}^{\mathrm{I}} + \delta_{ij}^{\mathrm{II}}$$

where:

$$\cos V_{ij}^{\circ} = \cos(v_i - v_j - \sigma_j + \sigma_i) ,$$

and:

$$\delta_{ij}^{\mathrm{I}} = -\tfrac{1}{4}(\gamma_i^2 + \gamma_j^2) \cos(v_i - v_j) + \tfrac{1}{4} \gamma_j^2 \cos(v_i + v_j - 2\vartheta_j) + \tfrac{1}{4} \gamma_i^2 \cos(v_i + v_j - 2\vartheta_i)$$
$$+ \tfrac{1}{2} \gamma_i \gamma_j \left\{ \cos(v_i - v_j - \vartheta_i + \vartheta_j) - \cos(v_i + v_j - \vartheta_i - \vartheta_j) \right\}$$

$$\delta_{ij}^{\mathrm{II}} = + \tfrac{1}{16}(\gamma_i^2 \gamma_j^2 - \gamma_j^4 - \gamma_i^4) \cos(v_i - v_j) - \tfrac{1}{16}(\gamma_i^2 \gamma_j^2 - \gamma_j^4) \cos(v_i + v_j - 2\vartheta_j)$$
$$- \tfrac{1}{16}(\gamma_i^2 \gamma_j^2 - \gamma_i^4) \cos(v_i + v_j - 2\vartheta_i) + \tfrac{1}{16} \gamma_i^2 \gamma_j^2 \cos(v_i - v_j - 2\vartheta_i + 2\vartheta_j) .$$

If we substitute $p = \gamma \cos \vartheta$, $q = -\gamma \sin \vartheta$, we find, neglecting the eccentricities and putting $m_i + m_j = \omega_{ij}$, $m_i - m_j = V_{ij}^{\circ}$:

$$\delta_{ij}^{\mathrm{I}} = -\tfrac{1}{4}(p_i^2 + p_j^2 + q_i^2 + q_j^2 - 2p_ip_j - 2q_iq_j) \cos V_{ij}^{\circ} + \tfrac{1}{4}(p_j^2 - q_j^2 + p_i^2 - q_i^2 - 2p_ip_j - 2q_iq_j) \cos \omega_{ij}$$
$$- \tfrac{1}{2}(p_iq_i + p_jq_j - p_iq_j - p_jq_i) \sin \omega_{ij} - \tfrac{1}{2}(p_jq_i - p_iq_j) \sin V_{ij}^{\circ}$$

$$\delta_{ij}^{\mathrm{II}} = + \tfrac{1}{16}(2p_i^2p_j^2 + 2q_i^2q_j^2 - 2p_i^2q_i^2 - 2p_j^2q_j^2 + 4p_ip_jq_iq_j - p_j^4 - p_i^4 - q_i^4 - q_j^4) \cos V_{ij}^{\circ}$$
$$- \tfrac{1}{16}(p_j^2p_iq_i - p_iq_iq_j^2 + p_i^2p_iq_j - p_ip_jq_i^2) \sin V_{ij}^{\circ} + \tfrac{1}{16}(p_i^4 - 2p_i^2p_j^2 + p_j^4 - q_i^4 + 2q_i^2q_j^2 - q_j^4) \cos \omega_{ij}$$
$$+ \tfrac{1}{8}(p_i^2p_jq_j + p_i^3q_i - p_j^3q_j - p_j^2p_iq_i + p_jq_jq_i^2 + p_iq_i^3 - p_iq_iq_j^2 - p_jq_j^3) \sin \omega_{ij}$$

From these expressions we conclude that, within our limits of approximation there are neither secular nor critical terms in the second part of III.

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*) Astron. Journ. April 1913 n° 646, 647 p. 179. It should be noticed that Hill's q_i is not the same as our q_i.

B). *Development of the principal part* $\frac{m_j}{\Delta_{ij}}$.

As is customary we develop: $\frac{m_j a_j}{\Delta_{ij}}$, where we suppose $a_j > a_i$, and denote the larger of the two by a', in order to avoid ambiguity. We then have the well-known formula:

$$\frac{a'}{\Delta_{ij}} = \frac{1}{2}\sum_{p=-\infty}^{p=+\infty} \beta_p \cos p\, V_{ij} = \beta_0 + \sum_{p=1}^{p=\infty} \beta_p \cos p\, V_{ij}.$$

In order to get a development in powers of the inclinations, we employ the formulae, given by *Hill* on pp. 179 and 180 of the above mentioned paper. Denoting the part of $\cos p V_{ij}$ that contains inclinations by $\Delta\, p V_{ij}$, we have:

$$\Delta \cos p V_{ij} = 2p\,\delta\left[\cos(p-1)\,V + \cos(p-3)\,V + \ldots\right]$$
$$+ 2p\,\delta^2\left[(p-1)\cos(p-2)\,V + 2(p-2)\cos(p-4)\,V + 3(p-3)\cos(p-6)\,V + \ldots\right] \tag{25}$$

where $\delta = \delta_{ij}^{\mathrm{I}} + \delta_{ij}^{\mathrm{II}}$, and only positive multiples of V must be taken; moreover $\cos 0\,V = \frac{1}{2}$ and:

$$\delta^2 = \left(\delta_{ij}^{\mathrm{I}}\right)^2 = \tfrac{1}{16}(\gamma_i^4 + 5\gamma_j^2\gamma_j^2 + \gamma_j^4) + \tfrac{1}{32}(\gamma_i^4 + 2\gamma_i^2\gamma_j^2 + \gamma_j^4)\cos 2V_{ij}^0 + \tfrac{1}{32}\gamma_j^4\cos(2\omega_{ij} - \vartheta_j)$$
$$+ \tfrac{1}{32}\gamma_i^4\cos(2\omega_{ij} - 4\vartheta_i) + \tfrac{1}{8}\gamma_i^2\gamma_j^2\cos(2V_{ij}^0 - 2\vartheta_i + 2\vartheta_j) + \tfrac{3}{16}\gamma_i^2\gamma_j^2\cos(2\omega_{ij} - 2\vartheta_i - 2\vartheta_j)$$
$$- \tfrac{1}{16}(\gamma_i^2 + \gamma_j^2)\gamma_j^2\cos 2(v_i - \vartheta_j) - \tfrac{1}{16}(5\gamma_i^2 + \gamma_j^2)\gamma_j^2\cos 2(v_j - \vartheta_j) - \tfrac{1}{16}(\gamma_i^2 + 5\gamma_j^2)\gamma_i^2\cos 2(v_i - \vartheta_i)$$
$$- \tfrac{1}{16}(\gamma_i^2 + \gamma_j^2)\gamma_j^2\cos 2(v_j - \vartheta_i) - \tfrac{1}{8}(\gamma_i^2 + \gamma_j^2)\gamma_i\gamma_j\cos(2V_{ij}^0 - \vartheta_i + \vartheta_j) - \tfrac{1}{4}(\gamma_i^2 + \gamma_j^2)\gamma_i\gamma_j\cos(\vartheta_i - \vartheta_j)$$
$$+ \tfrac{1}{8}(\gamma_i^2 + 2\gamma_j^2)\gamma_i\gamma_j\cos(2v_i - \vartheta_i - \vartheta_j) + \tfrac{1}{8}(2\gamma_i^2 + \gamma_j^2)\gamma_i\gamma_j\cos(2v_j - \vartheta_i - \vartheta_j) + \tfrac{1}{16}\gamma_i^2\gamma_j^2\cos 2(\vartheta_i - \vartheta_j)$$
$$+ \tfrac{1}{8}\gamma_i\gamma_j^3\cos(2v_j + \vartheta_i - 3\vartheta_j) - \tfrac{1}{8}\gamma_i\gamma_j^3\cos(2\omega_{ij} - \vartheta_i - 3\vartheta_j) + \tfrac{1}{8}\gamma_i^3\gamma_j\cos(2v_i - 3\vartheta_i + \vartheta_j)$$
$$- \tfrac{1}{8}\gamma_i^3\gamma_j\cos(2\omega_{ij} - 3\vartheta_i - \vartheta_j).$$

A complete development of $\frac{a'}{\Delta_{ij}}$ in terms of inclinations would be very long and complicated and of no importance for the further theory even when taking only terms of second degree in the inclinations. We therefore derive only the secular and critical terms.

a). *Secular terms in* III.

The eccentricities are again neglected which is allowable within our limit of accuracy. We thus have $\frac{r_i}{r_j} = \frac{a_i}{a_j}$, while β_p becomes equal to the Laplacean coefficient $b^{(p)}$. Now the part of $\frac{a'}{\Delta_{ij}}$ containing inclinations can be developed in multiples of V_{ij}^0, multiplied by δ and δ^2, and since δ and δ^2 contain only V_{ij}^0 and $2V_{ij}^0$, the following secular terms arise:

1° *from terms containing* $\cos^2 V_{ij}^0$.

The terms containing inclinations are:

$$\Delta_1 \frac{a'}{\Delta_{ij}} = \sum_{p=1}^{p=\infty} b^{(p)}\,\Delta\cos p V_{ij}$$

Gathering terms with the same argument $p\,V_{ij}^{\circ}$, we find, making use of the above expression of $\Delta \cos p V_{ij}$:

$$\Delta_1 \frac{a'}{\Delta_{ij}} = 2\,\delta \sum_{p=0}^{p=\infty} d^{(p)} \cos p\,V_{ij}^{\circ}$$

where:

$$d^{(p)} = (p+1)\,b^{(p+1)} + (p+3)\,b^{(p+3)} + (p+5)\,b^{(p+5)} + \ldots\ldots \text{ if } p > 0$$

and:

$$d^{(0)} = \tfrac{1}{2}\,[b^{(1)} + 3\,b^{(3)} + 5\,b^{(5)} + \ldots\ldots].$$

The term with $\cos^2 V_{ij}^{\circ}$ is:

$$2\,\delta\, d^{(1)} \cos V_{ij}^{\circ},$$

where δ is restricted to the part containing V_{ij}.

The secular part thus is:

$$d^{(1)}\Big[-\tfrac{1}{4}(p_i^2 + p_j^2 + q_i^2 + q_j^2 - 2\,p_ip_j - 2\,q_iq_j) + \tfrac{1}{16}(2\,p_i^2p_j^2 + 2\,q_i^2q_j^2 - 2\,p_i^2q_i^2 - 2\,p_j^2q_j^2 + 4\,p_ip_jq_iq_j - p_i^4 - p_j^4 - q_i^4 - q_j^4)\Big]$$

2° *from terms, containing* $\cos^2 2\,V_{ij}$.

These come from the part of the development of $\Delta \frac{a'}{\Delta_{ij}}$, which contains δ^2. Denoting it by $\Delta_2 \frac{a'}{\Delta_{ij}}$ we find:

$$\Delta_2 \frac{a'}{\Delta_{ij}} = 2\,\delta^2 \sum_{p=0}^{p=\infty} c^{(p)} \cos p\,V_{ij}^{\circ}$$

where:

$$c^{(p)} = (p+2)(p+1)\,b^{(p+2)} + 2\,(p+4)(p+2)\,b^{(p+4)} + 3\,(p+6)(p+3)\,b^{(p+6)} + \ldots\ldots \text{ if } p > 0$$

$$c^{(0)} = \tfrac{1}{2}\Big[2\,b^{(2)} + 2.4.2\,b^{(4)} + 3.6.3\,b^{(6)} + \ldots\ldots\Big].$$

Hence the terms with $\cos^2 2\,V_{ij}$ are:

$$2\,c^{(2)} \cos^2 2\,V_{ij}\Big[+\tfrac{1}{32}(\gamma_i^4 + 2\,\gamma_i^2\gamma_j^2 + \gamma_j^4) + \tfrac{1}{8}\gamma_i^2\gamma_j^2 \cos(2\,\vartheta_i - 2\,\vartheta_j) - \tfrac{1}{8}(\gamma_i^2 + \gamma_j^2)\,\gamma_i\gamma_j \cos(\vartheta_i - \vartheta_j)\Big]$$

Substituting the relations $p_i = \gamma_i \cos\vartheta_i$, $q_i = -\gamma_i \sin\vartheta_i$, we find for the secular term:

$$\tfrac{1}{32}\,c^{(2)}\Big[p_i^4 + 2\,p_i^2q_i^2 + q_i^4 + p_j^4 + 2\,p_j^2q_j^2 + q_j^4 + 6\,p_i^2p_j^2 - 2\,p_i^2q_j^2 - 2\,q_i^2p_j^2 + 6\,q_i^2q_j^2 + 16\,p_ip_jq_iq_j - 4\,p_i^3p_j - 4\,p_ip_j^3 - 4\,p_ip_jq_i^2 - 4\,p_ip_jq_j^2 - 4\,p_i^2q_iq_j - 4\,p_j^2q_iq_j - 4\,q_i^3q_j - 4\,q_j^3q_i\Big].$$

b). *Critical terms in* III.

The critical terms are those, which give rise to a small divisor. From the numerical values of the mean motions n_i it follows that:

$$2\,n_2 - n_1 \quad , \quad 2\,n_3 - n_2 \quad , \quad 4\,n_3 - n_1$$

are very small; hence critical terms can arise from the combinations of the satellites II—I, III—II and III—I.

For the sake of simplicity we neglect the eccentricities. We can divide the critical terms in two parts.

I. Critical terms arising from δ.

(1°). Combination i, j, where $i, j = 1,2$ or $2,3$.

The critical terms result from terms having the form:

$$\begin{matrix}\cos\\ \sin\end{matrix}\left(\alpha\, V_{ij} \pm \beta\, \omega_{ij}\right) = \begin{matrix}\cos\\ \sin\end{matrix}\left\{(\alpha \pm \beta)\, m_i - (\alpha \mp \beta)\, m_j\right\}$$

where $\alpha \mp \beta = 2(\alpha \pm \beta)$, and thus:

$$\alpha \pm 3\,\beta = 0.$$

Now δ only contains the argument ω_{ij}, thus $\beta = \pm 1$, whence a critical term arises only if $\alpha = 3$, viz:

$$2\, d^{(3)}\, \delta \cos 3\, V_{ij}^{\circ}.$$

The critical parts thus are:

$$d^{(3)}\Big[\tfrac{1}{16}\left(4p_j^2 - 4q_j^2 + 4p_i^2 - 4q_i^2 - 8p_ip_j - 8q_iq_j + p_i^4 - 2p_i^2p_j^2 + p_j^4 - q_i^4 + 2q_i^2q_j^2 - q_j^4\right)\cos(2m_i - 4m_j)$$
$$+ \tfrac{1}{8}\left(4p_iq_i + 4p_jq_j - 4p_iq_j - 4p_jq_i + p_i^2p_jq_j + p_i^3q_i - p_j^3q_j - p_j^2p_iq_i + p_jq_jq_i^2 + p_iq_i^3 - p_iq_iq_j^2 - p_jq_j^3\right)\sin(2m_i - 4m_j)\Big].$$

(2°). Combination i, j, where $i, j = 1,3$.

The terms belonging tho this combination are critical, if $\alpha \mp \beta = 4(\alpha \pm \beta)$, thus if:

$$3\,\alpha \pm 5\,\beta = 0$$

whence β must be a multiple of 3, which is not the case in terms of second or fourth order in the inclinations in δ.

II. Critical terms arising from δ^2.

Among the terms in δ^2 there occur

(1°). Terms of the form:

$$\begin{matrix}\cos\\ \sin\end{matrix}(\alpha V_{ij} \pm \beta\omega_{ij}),$$

which are critical if:

(A) $$\alpha \pm 3\,\beta = 0.$$

Now δ^2 only contains the argument $2\,\omega_{ij}$, whence α should be 6; the critical terms from this combination where j, $i = 2,1$ or $3,2$, are thus:

$$2\,\delta^2\, c^{(6)} \cos 6\, V_{ij}^{\circ} = c^{(6)}\Big[\tfrac{1}{32}\gamma_j^4 \cos 2(\omega_{ij} - 3\, V_{ij}^{\circ} - 2\,\vartheta_j) + \tfrac{1}{32}\gamma_i^4 \cos 2(\omega_{ij} - 3\, V_{ij}^{\circ} - 2\,\vartheta_i)$$
$$+ \tfrac{3}{16}\gamma_i^2\gamma_j^2 \cos 2(\omega_{ij} - 3\, V_{ij}^{\circ} - \vartheta_i - \vartheta_j) - \tfrac{1}{8}\gamma_i\gamma_j^3 \cos(2\,\omega_{ij} - 6\, V_{ij}^{\circ} - \vartheta_i - 3\,\vartheta_j)$$
$$- \tfrac{1}{8}\gamma_i^3\gamma_j \cos(2\,\omega_{ij} - 6\, V_{ij}^{\circ} - 3\,\vartheta_i - \vartheta_j)\Big]$$
$$= c^{(6)}\Big[\left(\tfrac{1}{32}p_i^4 + \tfrac{1}{32}q_i^4 + \tfrac{1}{32}p_j^4 + \tfrac{1}{32}q_j^4 - \tfrac{3}{16}p_i^2q_j^2 - \tfrac{3}{16}p_j^2q_i^2 - \tfrac{3}{16}p_i^2q_i^2 - \tfrac{3}{16}p_j^2q_j^2 + \tfrac{3}{16}p_i^2p_j^2\right.$$
$$+ \tfrac{3}{16}q_i^2q_j^2 - \tfrac{3}{4}p_ip_jq_iq_j - \tfrac{1}{8}p_ip_j^3 + \tfrac{3}{8}p_ip_jq_j^2 + \tfrac{3}{8}p_j^2q_iq_j - \tfrac{1}{8}q_iq_j^3 - \tfrac{1}{8}p_i^3p_j + \tfrac{3}{8}p_ip_jq_i^2$$
$$+ \tfrac{3}{8}p_i^2q_iq_j - \tfrac{1}{8}q_i^3q_j\Big)\cos 4(m_i - 2\,m_j) + \left(\tfrac{1}{8}p_jq_j^3 - \tfrac{1}{8}p_j^3q_j - \tfrac{1}{8}p_i^3q_i + \tfrac{1}{8}p_iq_i^3 + \tfrac{1}{8}p_j^3q_i\right.$$
$$- \tfrac{3}{8}p_iq_j^3 + \tfrac{1}{8}p_i^3q_j - \tfrac{3}{8}p_jq_i^3 - \tfrac{3}{8}p_ip_j^2q_i + \tfrac{3}{8}p_iq_iq_j^2 - \tfrac{3}{8}p_i^2p_jq_j + \tfrac{3}{8}p_jq_i^2q_j - \tfrac{3}{8}p_jq_iq_j^2$$
$$+ \tfrac{3}{8}p_ip_j^2q_j - \tfrac{3}{8}p_iq_jq_i^2 + \tfrac{3}{8}p_i^2p_jq_i\Big)\sin 4(m_i - 2\,m_j)\Big].$$

(B) $$3\alpha \pm 5\beta = 0$$

whence β must be divisible by 3, which is not the case.

(2°) Terms of the form:

$$\begin{matrix}\cos\\ \sin\end{matrix} (\alpha V_{ij}^{\circ} \pm \beta v_j).$$

These are critical if:

(A) $$\alpha \pm \beta = 0$$

In δ^2 there are only terms with $2v_j$, and thus $\alpha = 2$, whence the critical terms are:

$$c^{(2)}\Big[-\tfrac{1}{16}(\gamma_i^2+\gamma_j^2)\gamma_i^2\cos 2(v_j - V_{ij}^{\circ} - \vartheta_i) - \tfrac{1}{16}(5\gamma_i^2+\gamma_j^2)\gamma_j^2\cos 2(v_j - V_{ij}^{\circ} - \vartheta_j)$$
$$+\tfrac{1}{8}(2\gamma_i^2+\gamma_j^2)\gamma_i\gamma_j\cos(2v_j - 2V_{ij}^{\circ} - \vartheta_i - \vartheta_j) + \tfrac{1}{8}\gamma_i\gamma_j^3\cos(2v_j - 2V_{ij}^{\circ} + \vartheta_i - 3\vartheta_j)\Big]$$

$$= c^{(2)}\Big[\Big\{-\tfrac{1}{16}(p_i^2+p_j^2+q_i^2+q_j^2)(p_i^2-q_i^2) - \tfrac{1}{16}(5p_i^2+p_j^2+5q_i^2+q_j^2)(p_j^2-q_j^2)$$
$$+\tfrac{1}{8}(2p_i^2+2q_i^2+p_j^2+q_j^2)(p_ip_j+q_iq_j) + \tfrac{1}{8}(p_ip_j^3 - 3p_ip_jq_j^2 - 3p_j^2q_iq_j + q_iq_j^3)\Big\}\cos(2m_i - 4m_j)$$
$$+\Big\{\tfrac{1}{8}(p_i^2+p_j^2+q_i^2+q_j^2)p_iq_i + \tfrac{1}{8}(5p_i^2+p_j^2+5q_i^2+q_j^2)p_jq_j - \tfrac{1}{8}(2p_i^2+2q_i^2+p_j^2+q_j^2)(p_jq_i+p_iq_j)$$
$$+\tfrac{1}{8}(p_j^3q_i - 3p_iq_iq_j^2 - 3p_ip_j^2q_i + p_iq_j^3)\Big\}\sin(2m_i - 4m_j)\Big].$$

(B) $$3\alpha \pm \beta = 0,$$

thus β must be divisible by 3, which is not the case.

(3°) Terms of the form:

$$\begin{matrix}\cos\\ \sin\end{matrix} (\alpha V_{ij}^{\circ} \pm \gamma v_i).$$

These are critical if:

(A) $$\alpha \pm 2\gamma = 0.$$

In δ^2 $\gamma = 2$, whence α is 4; the critical terms are therefore:

$$c^{(4)}\Big[-\tfrac{1}{16}(\gamma_i^2+\gamma_j^2)\gamma_j^2\cos 2(v_i - 2V_{ij}^{\circ} - \vartheta_j) - \tfrac{1}{16}(5\gamma_j^2+\gamma_i^2)\gamma_i^2\cos 2(v_i - 2V_{ij}^{\circ} - \vartheta_i)$$
$$+\tfrac{1}{8}(\gamma_i^2+2\gamma_j^2)\gamma_i\gamma_j\cos(2v_i - 4V_{ij}^{\circ} - \vartheta_i - \vartheta_j) + \tfrac{1}{8}\gamma_i^3\gamma_j\cos(2v_i - 4V_{ij}^{\circ} - 3\vartheta_i + \vartheta_j)\Big]$$

$$= c^{(4)}\Big[\Big\{-\tfrac{1}{16}(p_i^2+p_j^2+q_i^2+q_j^2)(p_j^2-q_j^2) - \tfrac{1}{16}(5p_j^2+p_i^2+5q_j^2+q_i^2)(p_i^2-q_i^2)$$
$$+\tfrac{1}{8}(p_i^2+q_i^2+2p_j^2+2q_j^2)(p_ip_j+q_iq_j) + \tfrac{1}{8}(p_i^3p_j - 3p_ip_jq_i^2 - 3p_i^2q_iq_j + q_i^3q_j)\Big\}\cos(2m_i - 4m_j)$$
$$+\Big\{-\tfrac{1}{8}(p_i^2+p_j^2+q_i^2+q_j^2)p_jq_j - \tfrac{1}{8}(5p_j^2+p_i^2+5q_j^2+q_i^2)p_iq_i + \tfrac{1}{8}(p_i^2+q_i^2+2p_j^2+2q_j^2)(p_jq_i+p_iq_j)$$
$$-\tfrac{1}{8}(p_i^3q_j - 3p_iq_i^2q_j - 3p_i^2p_jq_j + p_jq_i^3)\Big\}\sin(2m_i - 4m_j)\Big].$$

(B) $$3\alpha \pm 4\gamma = 0$$

whence γ must be a multiple of 3, which case does not occur in δ^2.

The developments of III are strictly formal, but whether the process is legitimate has not yet been investigated. Without insisting on this point (see chapter V) we shall prove the convergency of the coefficients $d_{ij}^{(1)}$ and $c_{ij}^{(2)}$, which are to be used in the numerical integration of the differential equations.

The general formula for $d_{ij}^{(1)}$ is:

$$d_{ij}^{(1)} = 2\,b^{(2)} + 4\,b^{(4)} + 6\,b^{(6)} + \ldots\ldots + 2n\,b^{(2n)} + \ldots\ldots \text{ to infinity},$$

The general expression for the Laplacean coefficients is:

$$(26)\quad \frac{1}{2}b_s^{(i)} = \frac{s(s+1)\ldots(s+i-1)}{1.2\ldots i}\alpha^i\left[1 + \frac{s}{1}\cdot\frac{s+i}{i+1}\alpha^2 + \frac{s(s+1)}{1.2}\cdot\frac{(s+i)(s+i+1)}{(i+1)(i+2)}\alpha^4 + \ldots\right].$$

If $s = \frac{1}{2}$, we have:

$$b_{\frac{1}{2}}^{(i)} = b^{(i)}.$$

Formula (26) shows that $b^{(i)}$ is necessarily positive, whence $d_{ij}^{(1)}$ is an infinite series of positive terms. If we determine the ratio of the nth terms in $b_s^{(i)}$ and $b_s^{(i+2)}$, we find, denoting these terms by u_n and u_n':

$$\frac{u_n'}{u_n} = \frac{(s+i+n)(s+i+n+1)}{(i+n+1)(i+n+2)}\alpha^2.$$

If $s = \frac{1}{2}$, this becomes:

$$\frac{u_n'}{u_n} = \frac{1+2i+2n}{2+2i+2n}\cdot\frac{3+2i+2n}{4+2i+2n}\alpha^2,$$

whence:

$$\frac{u_n'}{u_n} < \alpha^2.$$

If now $\alpha < 1$ we may conclude: each term of $b^{(i)} >$ corresponding term of $b^{(i+2)}$ and, as $b^{(i)}$ and $b^{(i+2)}$ consist entirely of positive terms, we have:

$$b^{(2n+2)} < b^{(2n)}.$$

Denoting the general term of $d_{ij}^{(1)}$ by U_n, we immediately find:

$$\operatorname*{Lim}_{n=\infty}\frac{U_{n+1}}{U_n} = \operatorname*{Lim}_{n=\infty}\frac{2n+2}{2n}\,\frac{b^{(2n+2)}}{b^{2n}}$$

and this limit is obviously < 1.

In the case of Jupiter's satellites the largest value of α is about 0.63 and thus $d_{ij}^{(1)}$ in the theory of the satellites, converges. At the same time it is natural to conclude that at the beginning the series for $d_{ij}^{(1)}$ decreases very slowly, while the ratio for larger values of n approaches to α^2.

By the same reasoning the convergence of the series for $c_{ij}^{(2)}$ can be investigated and it is easy to verify that $c_{ij}^{(2)}$ also converges if $\alpha < 1$.

CHAPTER III.

INTEGRATION OF THE DIFFERENTIAL EQUATIONS NEGLECTING TERMS OF THE THIRD DEGREE IN THE INCLINATIONS.

The equations (F_1) will be integrated first analytically and afterwards numerically by the method of variation of arbitrary constants. It will be found that the integration can be performed with two systems of variables, which after the integration are to be compared. It is, however, desirable first to give some particulars concerning the units and constants employed, as they are different from those used by Souillart in the numerical evaluation of his analytical theory.

1. *The unit of time and the mean motions.*

In accordance with the theory of the longitudes we introduce instead of the time t a new independent variable defined by:

$$\tau = (n_2 - n_3)\, t$$

The mean motions in a mean solar day are according to de Sitter *)

$$\begin{aligned} n_1 &= 203^\circ\,4889\,9280 \\ n_2 &= 101^\circ\,3747\,6180 \\ n_3 &= 50^\circ\,3176\,4630 \\ n_4 &= 21^\circ\,5711\,1020 \end{aligned}$$

These mean motions are referred to the first point of Aries, therefore in order to get siderial mean motions they must be diminished by the precession, the amount of which is: 0°0000 3822. Further we introduce a small quantity $\varkappa$ by the relation: $\varkappa = \frac{n_2 - 2n_3}{n_2 - n_3}$. We have $n_2 - n_3 = 51^\circ 0571\,1550$. We measure the arcs in radians. The unit of time then is $1^d 1221\,899$ (its logarithm is: 0·0500 6634).

From the data of n_i it can be verified that $n_1 - 3n_2 + 2n_3 = 0$ exactly. From this relation and the definition of $\varkappa$, we easily conclude that the siderial mean motions in the new unit of time are:

$$n_1 = 4 - \varkappa, \qquad n_2 = 2 - \varkappa, \qquad n_3 = 1 - \varkappa, \qquad n_4 = c_4 - \varkappa$$

where

$$c_4 = 0{\cdot}4369\,7298 \text{ and } \varkappa = 0{\cdot}0144\,83925.$$

*) These annals Vol. XII, Part. I, p. 52.

2. *The unit of mass and the masses of the satellites.*

As unit of mass that of Jupiter is taken. The adopted masses are: *)

$$m_1 = 0{\cdot}0000\ 390$$
$$m_2 = 0{\cdot}0000\ 247^5$$
$$m_3 = 0{\cdot}0000\ 820$$
$$m_4 = 0{\cdot}0000\ 452^5$$

For the mass of the system we adopt the value of 1908 **) viz: $\frac{1}{1047{\cdot}40}$ astronomical units of mass. The mass of Jupiter is 1, hence the mass of the system is $1 + \Sigma\, m_i$.

Consequently the mass of the Sun is $m_s = 1047{\cdot}40\ (1 + \Sigma\, m_i) = 1047{\cdot}6000534$, for which we can take:

$$m_s = 1047{\cdot}60.$$

3. *The unit of length.*

In astronomical units $fm_0 = \frac{f}{1047{\cdot}60} = 0{\cdot}0000\ 0028\ 246679$, f being the Gaussian constant of attraction ($\log f = 8{\cdot}2355\ 8144 - 10$). The unit of length is defined so as to make $fm_0 = 1$. In order to express it in astronomical units, we have, calling it a (in astr. units) and the unit of time t (also in astr. units):

$$fm_0 = a^3\ t^{-2} = 0{\cdot}0000\ 0028\ 2466\ 79, \text{ whence:}$$
$$a = \sqrt[3]{0{\cdot}0000\ 0028\ 2466\ 79} \times 1{\cdot}1221899^2 = 0{\cdot}0070\ 854378.$$

4. *The mean distances.*

In deriving the equations of motion of a satellite we start from isoprotometric elements. As has been briefly pointed out the advantage of this system is that the mean distances are constants and at our disposal. The definition of the mean distances, chosen here, is based on the following consideration: In the ordinary elliptical motion the differential equations of the motion of a satellite would be:

$$\frac{d^2x_i}{dt^2} = -\,fm_0\,(1 + m_i)\,\frac{x_i}{r_i^3} = -\,a_i^3\,n_i^2\,\frac{x_i}{r_i^3}$$

according to Kepler's third law, with similar equations for the other co-ordinates.

Here the compression of the planet has been neglected, which, however, is not allowable. We therefore define a_i so as to take it into account. If we now neglect the inclinations of the orbits to Jupiter's equator and thus regard the motion as taking place in a plane, the equations of motion are:

$$\frac{d^2x_i}{dt^2} = -\,fm_0\,(1 + m_i)\,\frac{x_i}{r_i^3}\left(1 + \frac{fb^2}{r_i^2} + \frac{Kb^4}{2r_i^4} + \ldots\ldots\right) \tag{27}$$

with similar equations for the other co-ordinates.

Neglecting, as a sufficient approximation, periodical parts on the right-hand side of (27), we find from well-known developments:

$$\frac{d^2x_i}{dt^2} = -\,fm_0\,(1 + m_i)\left[1 + f_i\left(1 + \tfrac{1}{2}e_i^2 + \tfrac{3}{8}e_i^4 + \ldots\right) + K_i\left(\tfrac{1}{2} + \tfrac{3}{2}e_i^2 + \ldots\right)\right]\frac{x_i}{r_i^3}, \text{ etc.} \tag{28}$$

*) Cf. These annals Vol XII, 1 p. 49 footnote and Cape XII,5, p. 13 footnote.

**) Elements and Masses p. 717.

where:

$$J_i = \frac{Jb^2}{a_i^2} \quad , \quad K_i = \frac{Kb^4}{a_i^4} .$$

The orbit, which the satellite would describe in consequence of (28), is an ellipse and according to Kepler's third law we should have:

$$a_i^3 n_i^2 = \mu_i \left[1 + J_i \left(1 + \tfrac{1}{2} e_i^2 + \ldots\right) + K_i \left(\tfrac{1}{2} + \tfrac{3}{2} e_i^2 + \ldots\right)\right] .$$

Neglecting the terms $\frac{1}{2} J_i e_i^2$, $\frac{3}{2} K_i e_i^2$, etc., the semi-axis major a_i is defined by:

$$a_i^3 n_i^2 = \mu_i \left[1 + J_i + \tfrac{1}{2} K_i \right] . \tag{29}$$

The value of a_i from (29) is the mean distance of the satellite, employed throughout the following. The numerical values for the four satellites then are:

$$\begin{aligned} \log a_1 &= 9{\cdot}5997\ 7377 - 10 \\ \log a_2 &= 9{\cdot}8014\ 5730 - 10 \\ \log a_3 &= 0{\cdot}0042\ 5020 \\ \log a_4 &= 0\ 2494\ 6693 \end{aligned}$$

while for the mean distance a_s of the Sun, we adopt: $\log a_s = 2{\cdot}865871$.

5. *Eccentricities.*

For the eccentricities we adopt the following values:

$$\begin{aligned} e_1 &= 0{\cdot}00404 \\ e_2 &= 0{\cdot}00937 \\ e_3 &= 0{\cdot}00175 \\ e_4 &= 0{\cdot}00745 \\ e_s &= 0{\cdot}04825 \end{aligned}$$

As to the eccentricities of the first three satellites, they have been determined, while regarding the large inequalities x_i, which arise as perturbations of the eccentricities, as eccentricities and taking as the average value of e_i that defined by:

$$e_i^2 = x_i^2 + \Sigma_j (\tau_{ij} e_j)^2 ,$$

where τ_{ij} are the coefficients in the equations of the centre and e_j are the proper eccentricities, for which we adopt de Sitter's data *), viz:

$$e_1 = 0°0031 \pm 0°0080 \ , \ e_2 = 0°0172 \pm 0°0040 \ , \ e_3 = 0°0868 \pm 0°0065 \ , \ e_4 = 0°4264 \pm 0°0020.$$

The meaning of τ_{ij} is thus the same as in:

$$\begin{aligned} 2\ e_i \sin \varpi_i &= 2\ \Sigma_j\ \tau_{ij}\ e_j \sin \varpi_j \\ 2\ e_i \cos \varpi_i &= 2\ \Sigma_j\ \tau_{ij}\ e_j \cos \varpi_j \end{aligned}$$

from which in the theory of the longitudes the perijoves and eccentricities are computed.

*) Elements and Masses p. 596.

For x_i we have used:

$$x_1 = 0{\cdot}00404 \quad , \quad x_2 = 0{\cdot}00936 \quad ; \quad x_3 = 0{\cdot}00060.$$

These are the values which, according to an unpublished computation by de Sitter, correspond to the above adopted masses. The inequalities of the fourth satellite being negligible, e_4 is computed from the formula: $e_4^2 = \Sigma_j (\tau_{4j}\, e_j)^2$. Employing with the same definition of the e_i the formulae of de Sitter *), with, — according to the masses and value of Jb^2, adopted by us, —: $\lambda_1 = 0{\cdot}500$, $\lambda_2 = +\ 0{\cdot}0714$, $\lambda_3 = +\ 0{\cdot}0199$, $\rho = -\ 0{\cdot}03506$, we find values slightly differing from those adopted above, viz: 0·00401, 0·00931, 0·00059.

6. *Quantities connected with the compression of the planet.*

The quantities relating to the compression of Jupiter are Jb^2 and Kb^4. It may be observed that the numerical values of J and K are independent of the particular units of length and mass, which are employed, but that they always occur multiplied by b^2 and b^4 respectively, where b is the equatorial radius of the planet.

The value of Jb^2 which we adopt is, expressed in astronomical units, $Jb^2 = 0{\cdot}0000\ 0000\ 500$. **) For b we adopt the value of Sampson, who finds for the apparent equatorial radius of Jupiter at its mean distance from the earth 18″930. ***) This gives $\log b = 6{\cdot}678\,961$ and thus $J = 0{\cdot}021\,9305$, a value quite different from Souillart's, who uses 0·024 7187.

Expressed in our unit of length we thus find: $\log Jb^2 = 5{\cdot}998\,2366 - 10$.

For K we adopt the value $K = 0{\cdot}00217$, derived by de Sitter. **) We then have $\log Kb^4 = 2{\cdot}650\,600 - 10$

The coefficients of the differential equations contain C, the principal moment of inertia of Jupiter with respect to its axis of revolution. The value of C can be derived, if Jb^2 and the quantity $H = \dfrac{C - A}{C}$ are known, where A is the principal moment of inertia with respect to the equatorial diameter $2b$.

We have put:

$$m_0 Jb^2 = \frac{2}{5}\pi \int_0^b \rho \frac{d\,[(b^2 - a^2)\ b^2 a]}{db}\, db$$

In the case of an ellipsoid of revolution the principal moments of inertia are expressed by:

$$C = \frac{8}{15}\pi \int_0^b \rho \frac{d\,(b^4 a)}{db}\, db$$

$$A = \frac{4}{15}\pi \int_0^b \rho \frac{d\left\{b^2 a\,(b^2 + a^2)\right\}}{db}\, db\,,$$

whence we have:

$$m_0 Jb^2 = \frac{3}{2}(C - A)\,.$$

Now the value of Jb^2 is known. The value of H has been derived by de Sitter from the compression

*) Elements and Masses p. 720. It should be noticed that the x_i there given is twice the quantity here denoted by x_i, and is expressed in degrees instead of in radians.

**) See Leiden XII, 1, p. 49 footnote.

***) See de Sitter "On the formulas...." *M. N.* 1910 p. 96.

of the planet and the value of J by means of Clairaut's theory. *) He finds $H = 0\cdot0602$, but this value is still very uncertain. The corresponding value of C is:

$$\log C = 7\cdot042550 - 10.$$

Analytical integration of the differential equations.

The differential equations (F_1), (F_2), when neglecting terms of the third degree in the inclinations and introducing the new variable $\tau = l_2 - l_3$, are as follows:

$$(F_{11}) \quad \begin{cases} \dfrac{dP_i}{d\tau} = \dfrac{1}{\sqrt{G_i}} \dfrac{\partial R_i}{\partial q_i} + \dfrac{\sqrt{G_i}}{Cn} \Sigma_j \dfrac{\partial U_j}{\partial q_j} \\ \dfrac{dQ_i}{d\tau} = -\dfrac{1}{\sqrt{G_i}} \dfrac{\partial R_i}{dp_i} - \dfrac{\sqrt{G_i}}{Cn} \Sigma_j \dfrac{\partial U_j}{\partial p_j} \\ \qquad i = 1, \ldots 4, \; j = 1, \ldots 4, 5. \\ \dfrac{d\alpha}{d\tau} = -\dfrac{1}{Cn} \Sigma_\lambda \dfrac{\partial U_\lambda}{\partial p_\lambda} \\ \dfrac{d\beta}{d\tau} = +\dfrac{1}{Cn} \Sigma_\lambda \dfrac{\partial U_\lambda}{\partial q_\lambda} \end{cases}$$

and:

$$(F_{22}): \quad \frac{d\sigma_i}{d\tau} = -\frac{1}{2G_i}\left(p_i \frac{\partial R_i}{\partial p_i} + q_i \frac{\partial R_i}{\partial q_i}\right), \; i = 1, \ldots 4.$$

The first system of equations is a set of linear differential equations, while the last equation is, even when neglecting R^{IV}, not linear. Therefore the first system will be treated apart and the values of P_i and Q_i, found from it will be substituted in the last equation, which can then easily be integrated.

This process is of course an approximation, which, however, is quite sufficient for our purpose. Besides, an exact integration of the whole system is impossible.

The right hand sides of the equations contain p_s and q_s, relating to the Sun. These depend on the variables α and β, the expressions for which are derived below and will be found to be very simple. This suggests introducing as variables p_s and q_s, instead of α and β, from which after the integration α and β are to be derived. The method in which this is done will be called method II, while in method I the original variables α and β are retained. Thus the variables are:

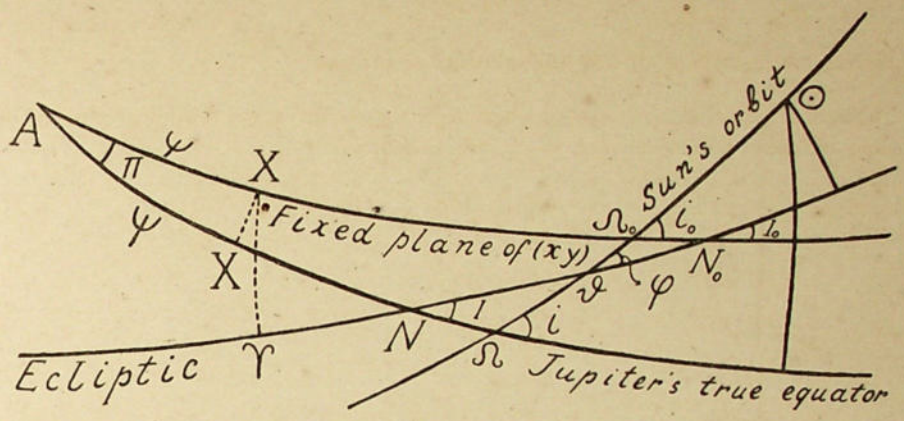
$$\text{Method I} \begin{cases} P_1, \ldots P_4, P_5 = \beta \cos i_0 \\ Q_1, \ldots Q_4, Q_5 = \alpha \cos i_0 \end{cases}$$

$$\text{Method II} \begin{cases} P_1, \ldots P_4, P_5 = p_s \\ Q_1, \ldots Q_4, Q_5 = q_s \end{cases}$$

In order to derive the relation between p_s, q_s and α, β a transformation of co-ordinates must be applied. Now α and β occur only in connection with i_0 and $\mathfrak{N}_0$, which refer to the fixed plane of (x, y), while p_s and q_s refer to the moving true equator of Υ. The theory of secular perturbations of the major

*) These annals Vol XII, 1, p. 50.

planets furnishes the inclination and node of the Sun's orbit to the ecliptic as a plane of reference and from these data the inclination and node with reference to the chosen plane of (x, y) must be derived. It is clear that we need therefore a double transformation.



(The axis of X is so chosen that $N_0 \Upsilon = N_0 X$.)

[It should be noticed that the motions are considered relatively to $\mathfrak{Y}$, so that what is called in the figure the Sun's orbit, actually is the orbit of Jupiter, etc.]

The inclinations and nodes (in the following node = ascending node) of the Sun's orbit referred to the ecliptic being denoted by φ and ϑ, to the fixed plane of (x, y) by i_0 and Ω_0, those of the ecliptic to the same plane of (x, y) by I_0 and N_0, we introduce:

$$\begin{aligned} \mathcal{P} &= \sin I_0 \cos N_0 & p_{s0} &= \sin i_0 \cos \Omega_0 & P_s' &= \sin \varphi \cos \vartheta \\ \mathcal{Q} &= -\sin I_0 \sin N_0 & q_{s0} &= -\sin i_0 \sin \Omega_0 & Q_s' &= -\sin \varphi \sin \vartheta \end{aligned}$$

We then have from the spherical triangle $N_0 \Omega_0 \vartheta$:

$$\sin (N_0 - \vartheta) \sin \varphi = \sin (N_0 - \Omega_0) \sin i_0$$

$$\sin I_0 \cos \varphi + \cos (N_0 - \vartheta) \sin \varphi \cos I_0 = \cos (N_0 - \Omega_0) \sin i_0$$

from which after some reductions, we find:

$$-\sin \Omega_0 \sin i_0 = -\sin \vartheta \sin \varphi - \sin I_0 \sin N_0 \left[\cos \varphi - \tan \tfrac{1}{2} I_0 \sin \varphi \cos (\vartheta - N_0)\right]$$

$$\cos \Omega_0 \sin i_0 = \cos \vartheta \sin \varphi + \sin I_0 \cos N_0 \left[\cos \varphi - \tan \tfrac{1}{2} I_0 \sin \varphi \cos (\vartheta - N_0)\right]$$

or:

$$\begin{aligned} p_{s0} &= P_s' + a\mathcal{P} \\ q_{s0} &= Q_s' + a\mathcal{Q}, \end{aligned}$$

where

$$a = \cos \varphi - \tan \tfrac{1}{2} I_0 \sin \varphi \cos (\vartheta - N_0)$$

The values of φ and ϑ, that occur in a are not constant; their variation with the time, especially that of φ is, however, very slight, and we can practically treat a as a constant. [For φ and ϑ de Sitter gives in *"Elements and Masses"* p. 722 $\varphi = 1° 18' 31''1 - 0''2051\ T$, $\vartheta = 99° 26' 36'' + 36''396\ T$, where T is expressed in tropical years, reckoned from 1900 Jan. 0·0 M. T. Greenwich].

If now we put:

$$\begin{aligned} \cos\vartheta \sin\varphi &= p + at \\ -\sin\vartheta \sin\varphi &= q + bt, \end{aligned} \tag{30}$$

where t is expressed in mean solar days, we also have:

$$\begin{aligned} p_{so} &= p_{so}' + at \\ q_{so} &= q_{so}' + bt \end{aligned}$$

where a and b are the same as in (30) and p_{so}' and q_{so}' are the values at the initial epoch, for which we take 1900.0. For the fixed plane of (x, y), we have chosen Jupiter's mean equator at the epoch 1900·0. The inclination and node of the orbital plane with reference to this plane, are then determined by the formulae ("*Elements and Masses*" p. 723):

$$\begin{aligned} i_o &= 3^\circ\ 6'\ 55''1 + 0''0243\ T \\ \Omega_o &= 135^\circ\ 46'\ 44''29 + 50''155\ T, \text{ from } 1900\cdot0. \end{aligned} \tag{31}$$

If we denote the values at the initial epoch by i_o' and Ω_o', we find:

$$\begin{aligned} p_{so}' &= \sin i_o' \cos\Omega_o' = -0{\cdot}038\,947 \\ q_{so}' &= -\sin i_o' \sin\Omega_o' = -0{\cdot}037\,902\,. \end{aligned}$$

Souillart II gives (p. 165) for b and a $-0''09475$ and $+0''13008$ resp.
Tisserand IV gives (p. 57) $-0''09557$ „ $+0''12949$ per Julian year.

Expressed in our units these constants become:

Souillart II	$-0{\cdot}000\,000\,001\,937$	$+0{\cdot}000\,000\,001\,412$
Tisserand IV	$-0{\cdot}000\,000\,001\,929$	$+0{\cdot}000\,000\,001\,423$.

The values of Souillart are taken from Leverrier's theory ot the major planets. Tisserand does not quote his authority, but it is very probable that his values are of more recent date and better. We adopt the latter and thus find:

$$\begin{aligned} p_{so} &= -0{\cdot}038\,947 - 0{\cdot}000\,000\,001\,929\ \tau \\ q_{so} &= -0{\cdot}037\,902 + 0\ 000\,000\,001\,423\ \tau, \end{aligned}$$

Expressing the co-ordinates p_s and q_s*) in p_{so} and q_{so} by means of the spherical triangle $A\ \Omega\ \Omega_o$, we find:

$$p_s = p_{so} + \beta\left[\cos i_o - \sin i_o \tan\frac{\pi}{2}\cos(\Omega_o + \psi)\right] = p_{so} + \beta\cos i_o,$$

$$q_s = q_{so} + \alpha\left[\cos i_o - \sin i_o \tan\frac{\pi}{2}\cos(\Omega_o + \psi)\right] = q_{so} + \alpha\cos i_o,$$

neglecting quantities of an order higher than π^2.

Therefore we may put, substituting the values of p_{so} and q_{so}:

$$\begin{aligned} p_s &= p_{so}' + a'\tau + \beta\cos i_o \\ q_s &= q_{so}' + b'\tau + \alpha\cos i_o \end{aligned}$$

*) The meaning of p_s and q_s is: $p_s = \sin i\cos\Omega$
$q_s = -\sin i\sin\Omega$

where:

$$a' = -\ 0{\cdot}000\,000\,001\,937\ ^{*})$$
$$b' = +\ 0{\cdot}000\,000\,001\,412$$

whence the squares of a′ and b′ may, with our degree of accuracy, be entirely neglected.

The most important of the perturbations of the inclinations and nodes are the secular inequalities. In comparison with these the greater part of the periodic inequalities can be neglected, except those, whose coefficients become large on account of small denominators. The process to be followed in order to integrate the system (F_{11}) is to substitute on the right-hand sides the secular part of the perturbative function, neglecting other terms.

Gathering the secular terms of the perturbative function, we find:

$$S[R_i] = -\frac{J_i}{a_i}\mu_i\left[\tfrac{1}{2}(1+\tfrac{3}{2}e_i^2)(p_i^2+q_i^2)\right] - \frac{K_i}{a_i}\mu_i\left[\tfrac{1}{2}(1+5\,e_i^2)(p_i^2+q_i^2)\right]$$
$$+\sum_j \frac{m_j}{a'} d_{ij}^{(1)}\left[-\tfrac{1}{4}(p_i^2+p_j^2+q_i^2+q_j^2-2\,p_ip_j-2\,q_iq_j)\right].$$

where $j = 1, \ldots 4, s$; the value $j = i$ being excluded.

$$S[U_i] = -\frac{J_i}{a_i} m_i\left[\tfrac{1}{2}(1+\tfrac{3}{2}e_i^2)(p_i^2+q_i^2)\right] - \frac{K_i}{a_i} m_i\left[\tfrac{1}{2}(1+5\,e_i^2)(p_i^2+q_i^2)\right]$$

We now introduce the following abbreviations:

$$A_i = -\tfrac{1}{2}\frac{J_i}{a_i}\mu_i(1+\tfrac{3}{2}e_i^2) - \tfrac{1}{2}\frac{K_i}{a_i}\mu_i(1+5\,e_i^2)$$

$$C_i = -\tfrac{1}{2}\frac{J_i}{a_i}m_i(1+\tfrac{3}{2}e_i^2) - \tfrac{1}{2}\frac{K_i}{a_i}m_i(1+5e_i^2) = \frac{m_i}{\mu_i}A_i, \text{ whence } C_i \text{ is much smaller than } A_i$$

$$B_{ij} = -\tfrac{1}{4}\frac{m_j}{a'}d_{ij}^{(1)}$$

Hence:

$$S[R_i] = (A_i+\Sigma_j B_{ij})p_i^2 + (A_i+\Sigma_j B_{ij})q_i^2 + \Sigma_j B_{ij}p_j^2 + \Sigma_j B_{ij}q_j^2 - 2\Sigma_j B_{ij}(p_ip_j+q_iq_j)$$
$$S[U_i] = C_i(p_i^2+q_i^2)$$

from which we deduce:

$$\frac{\partial R_i}{\partial p_i} = 2(A_i+\Sigma_j B_{ij})p_i - 2\Sigma_j B_{ij}p_j$$
$$\frac{\partial R_i}{\partial q_i} = 2(A_i+\Sigma_j B_{ij})q_i - 2\Sigma_j B_{ij}q_j$$
$$\frac{\partial U_i}{\partial p_i} = 2C_ip_i \qquad (32)$$
$$\frac{\partial U_i}{\partial q_i} = 2C_iq_i$$

where for R_i and U_i we must take $S[R_i]$ and $S[U_i]$.

*) By an oversight we have used these values instead of Tisserand's in the numerical computations, but this does not essentially alter the results of the end of this chapter.

(A) *Integration according to method I.*

The variables are: $P_1, P_2, \ldots P_4, P_5 = \beta \cos i_o$; $Q_1, Q_2, Q_5 = \alpha \cos i_o$.

$\log \cos i_o = 9{\cdot}999\,3577 - 10$, and thus $\cos i_o$ is very nearly equal to unity.

The substitution of $\beta \cos i_o$ and $\alpha \cos i_o$ for β and α is made in order that the coefficients of the fifth variable may be the same in both methods.

Substituting the relations (32) in (F_{11}), and introducing the following symbols:

$$
\begin{aligned}
[i] &= \frac{2}{G_i}\,[A_i + \Sigma_j B_{ij}] + \frac{2\,C_i}{Cn}, \text{ where } j \neq i \\
(ij) &= -\frac{2\,B_{ij}}{\sqrt{G_i G_j}} + \frac{2\,C_j}{Cn}\sqrt{\frac{G_i}{G_j}}\,, \quad \text{,,} \quad j \neq i \\
(i5) &= -\frac{2\,B_{i5}}{\sqrt{G_i}} + \frac{2\,C_5}{Cn}\sqrt{G_i} \\
\{i\} &= \frac{2\,C_i}{Cn\sqrt{G_i}} \cos i_o, \; i = 1 \ldots 4 \\
\{5\} &= \frac{2\,C_5}{Cn} \cos i_o
\end{aligned}
$$

the system of equations becomes:

$$
\begin{aligned}
&\frac{dP_1}{d\tau} - [1]\,Q_1 - (12)\,Q_2 - (13)\,Q_3 - (14)\,Q_4 - (15)\,Q_5 = \quad (15)\,(q_{50}' + b'\tau) \\
&\frac{dQ_1}{d\tau} + [1]\,P_1 + (12)\,P_2 + (13)\,P_3 + (14)\,P_4 + (15)\,P_5 = -(15)\,(p_{50}' + a'\tau) \\
&\qquad\qquad \text{etc.} \qquad\qquad (F_3) \\
&\frac{dP_5}{d\tau} - \{1\}\,Q_1 - \{2\}\,Q_2 - \{3\}\,Q_3 - \{4\}\,Q_4 - \{5\}\,Q_5 = \quad \{5\}\,(q_{50}' + b'\tau) \\
&\frac{dQ_5}{d\tau} + \{1\}\,P_1 + \{2\}\,P_2 + \{3\}\,P_3 + \{4\}\,P_4 + \{5\}\,P_5 = -\{5\}\,(p_{50}' + a'\tau)
\end{aligned}
$$

The system (F_3) is a system of simultaneous linear differential equations with constant coefficients. Omitting at first the right hand sides of (F_3) the integrals of the reduced system are:

$$
\begin{aligned}
P_i &= \Sigma_j \sigma_{ij} N_j \cos(b_j \tau + M_j) = \Sigma_j \sigma_{ij} K_j \\
Q_i &= \Sigma_j \sigma_{ij} N_j \sin(b_j \tau + M_j) = \Sigma_j \sigma_{ij} L_j
\end{aligned}
\qquad (33)
$$

where

$$j = 1, \ldots 5, \text{ including } j = i.$$

N_j and M_j are constants of integration to be determined from the initial values P_o and Q_o, which are derived from observations, while σ_{ij} and b_j are constants depending on the masses and other elements. In order to determine σ_{ij} and b_j we substitute the values (33) in (F_3) and equating to zero the coefficient of the sine or cosine of the argument $b_j\tau + M_j$, we find:

$$
(G)\quad \left\{
\begin{aligned}
&\big[b_j + [1]\big]\,\sigma_{1j} + (12)\,\sigma_{2j} + (13)\,\sigma_{3j} + (14)\,\sigma_{4j} + (15)\,\sigma_{5j} = 0 \\
&(21)\,\sigma_{1j} + \big[b_j + [2]\big]\,\sigma_{2j} + (23)\,\sigma_{3j} + (24)\,\sigma_{4j} + (25)\,\sigma_{5j} = 0 \\
&\qquad\qquad \text{etc.} \\
&\{1\}\,\sigma_{1j} + \{2\}\,\sigma_{2j} + \{3\}\,\sigma_{3j} + \{4\}\,\sigma_{4j} + \big[b_j + \{5\}\big]\,\sigma_{5j} = 0
\end{aligned}
\right.
$$

This system of homogeneous linear equations only admits a set of values of σ_{ij}, if the determinant of the system is zero. This condition gives an equation of the fifth degree in b, with five real roots b_j. Among these $b_1, \ldots b_4$ are the most important and much larger than b_5, which relates to the motion of Jupiter's equator.

Souillart in his first method of integration does not solve this equation of the fifth degree, but as a first approximation neglects the last equation depending on the motion of the equator (See *Souillart* I p. 127). From this last equation he concludes that the value of $(b_j + \bigcirc) N^{IV}$ *) and thus also $b_j N^{IV}$ must be very small, and as the coefficients (0), (1), ... etc., do not differ much from the b_j, he considers it as obvious that $(0)N^{IV}$ etc. may be neglected, and so he finds a set of four equations, from which he computes the $b_1, \ldots b_4$ and the ratios of the N's. For the determination of the $b_1, \ldots, b_4$, this process may be accepted as valid, because the determinant, that gives the equation in b_j practically reduces to its diagonal, the other terms being very small compared with those of the diagonal. It is therefore probable that the values derived from this approximative method will not differ much from the actual values.

But for the determination of the ratios $\frac{N''}{N}$ etc. the method is not legitimate, for although the quantities $(0)N^{IV}$ etc. are small, they are quite comparable with the other quantities $(0,3)N'''$, etc. This evidently is the explanation of the fact that Souillart, when solving not the system of four equations but the complete system found nearly the same values for the b_j, while for the ratios $\frac{N''}{N}$ the deviation from the former values were more important, in one case the sign of $\frac{N''}{N}$ being reversed.

Consequently, when deriving with the values obtained the elliptical part of the latitudes he found discrepancies of several seconds of arc. Souillart himself in the second part of his theory has preferred the last method of integration (*Souillart* II p. 162), without giving any reasons.

When applying to our system (G) Souillart's method it is very probable that not only the values b_j determined according to both methods, but also the ratios $\frac{N''}{N}$ (our σ_{ij}) would show only slight discrepancies, for from the last equation (G) we may conclude that $b_j \sigma_{5j}$ is small; the largest value that it can assume is about 0·000 004 (see table on page 61) for $j = 1$; but then $b_j = b_1$ is the largest value of the b_j's, and about $10^3 \times$ the largest value of $(j5)$, so that undoubtedly the terms $(j5)\sigma_{5j}$ can be neglected with sufficient approximation.

The only rigorous method, however, is to consider the complete system (G). The equation of the fifth degree in b then becomes:

$$(H) \qquad \begin{vmatrix} b + [1] & (12) & (13) & (14) & (15) \\ (21) & b + [2] & (23) & (24) & (25) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \{1\} & \{2\} & \{3\} & \{4\} & b + \{5\} \end{vmatrix} = 0$$

To each of the solutions of equations (H) corresponds a system of equations (G) for determining the ratios of four of the σ_{ij} to the fifth. The latter then is an arbitrary constant. It is customary to take $\sigma_{jj} = 1$. Then four of the equations (G) suffice to determine the remaining σ_{ij}.

*) In Souillart's notation.

If now we integrate the complete equations (F_3) according to the method of variation of constants the integrals (33) retain the same form, but now N_j and M_j are variable. Substituting (33) in (F_3) we find, as (33) identically satisfy the equations without the right-hand side:

$$\Sigma_j\ \sigma_{1j}\ \frac{dN_j}{d\tau}\ \cos(b_j\tau + M_j) - \Sigma_j\ \sigma_{1j}\ N_j\ \frac{dM_j}{d\tau}\ \sin(b_j\tau + M_j) = \quad (15)\ (q_{so}' + \mathrm{b}'\tau)$$

(K) etc.

$$\Sigma_j\ \sigma_{1j}\ \frac{dN_j}{d\tau}\ \sin(b_j\tau + M_j) + \Sigma_j\ \sigma_{1j}\ N_j\ \frac{dM_j}{d\tau}\ \cos(b_j\tau + M_j) = -\ (15)\ (p_{so}' + \mathrm{a}'\tau)$$

etc.

We now introduce:

$$u_j = \frac{dN_j}{d\tau}\cos(b_j\tau + M_j) - N_j\frac{dM_j}{d\tau}\sin(b_j\tau + M_j)$$

$$v_j = \frac{dN_j}{d\tau}\sin(b_j\tau + M_j) + N_j\frac{dM_j}{d\tau}\cos(b_j\tau + M_j)$$

Further, since $\mathrm{b}'^3\tau^3$, $\mathrm{a}'^3\tau^3$ can be neglected, we put instead of $\mathrm{b}'\tau$ and $\mathrm{a}'\tau$ $\sin \mathrm{b}'\tau$ and $\sin \mathrm{a}'\tau$ respectively. Then u_j and v_j are determined by the equations:

$$\sigma_{11}u_1 + \sigma_{12}u_2 + \sigma_{13}u_3 + \sigma_{14}u_4 + \sigma_{15}u_5 = \quad (15)\ (q_{so}' + \sin \mathrm{b}'\tau)$$

(34) . . . . . .

$$\sigma_{51}u_1 + \sigma_{52}u_2 + \sigma_{53}u_3 + \sigma_{54}u_4 + \sigma_{55}u_5 = \quad \{5\}\ (q_{so}' + \sin \mathrm{b}'\tau)$$

and:

$$\sigma_{11}v_1 + \sigma_{12}v_2 + \sigma_{13}v_3 + \sigma_{14}v_4 + \sigma_{15}v_5 = -\ (15)\ (p_{so}' + \sin \mathrm{a}'\tau)$$

(35) . . . . . .

$$\sigma_{51}v_1 + \sigma_{52}v_2 + \sigma_{53}v_3 + \sigma_{54}v_4 + \sigma_{55}v_5 = -\ \{5\}\ (p_{so}' + \sin \mathrm{a}'\tau)$$

Thus u_j and v_j are of the form:

$$u_j = K_{1j} + K_{2j}\sin \mathrm{b}'\tau$$
$$v_j = K_{1j}^{\circ} - K_{2j}\sin \mathrm{a}'\tau$$

The advantage of the substitution $\sin \mathrm{a}'\tau$ and $\sin \mathrm{b}'\tau$ for $\mathrm{a}'\tau$ and $\mathrm{b}'\tau$ evidently is that τ does not enter explicitly into the expressions for u_j and v_j and consequently the corrections to N_j and M_j are periodic. From the equations (K) we find:

(36)
$$\frac{dN_j}{d\tau} = u_j\cos(b_j\tau + M_j) + v_j\sin(b_j\tau + M_j)$$
$$N_j\frac{dM_j}{d\tau} = -\ u_j\sin(b_j\tau + M_j) + v_j\cos(b_j\tau + M_j)$$

Hence the equation determining the correction δN_j is:

$$\frac{dN_j}{d\tau} = (K_{1j} + K_{2j}\sin \mathrm{b}'\tau)\cos(b_j\tau + M_j) + (K_{1j}^{\circ} - K_{2j}\sin \mathrm{a}'\tau)\sin(b_j\tau + M_j)$$

from which we see that the correction of N_j is purely periodic and may, on account of the very small values of K_{1j} be neglected. Therefore in the second equation (36) N_j can be regarded as a constant and it is then easily verified that the correction δM_j is also periodic. If thus we integrate the complete equations (F_3) by the method of variation of constants, the above development shows that the coefficient b_j of τ in the argument of the integrals (33) needs no correction.

It is desirable to investigate the alterations produced in P_i and Q_i by the corrections to N_j and M_j. Let the corrections to P_i and Q_i be denoted by δP_i and δQ_i, then it is possible to derive δP_i and δQ_i directly, for from (33) we find:

$$\begin{aligned}\delta P_i &= \Sigma_j\, \sigma_{ij} \left\{ \cos b_j\tau\ \delta\ (N_j \cos M_j) - \sin b_j\tau\ \delta\ (N_j \sin M_j) \right\} \\ \delta Q_i &= \Sigma_j\, \sigma_{ij} \left\{ \sin b_j\tau\ \delta\ (N_j \cos M_j) + \cos b_j\tau\ \delta\ (N_j \sin M_j) \right\},\end{aligned} \tag{37}$$

$$j = 1, \ldots 4, 5.$$

Now from (36) we have:

$$\begin{aligned}\frac{d\ (N_j \cos M_j)}{d\tau} &= \quad u_j \cos b_j\tau + v_j \sin b_j\tau \\ \frac{d\ (N_j \sin M_j)}{d\tau} &= -\ u_j \sin b_j\tau + v_j \cos b_j\tau\end{aligned}$$

Integrating these two equations, we find:

$$\begin{aligned}\delta (N_j \cos M_j) &= \frac{K_{1j}}{b_j} \sin b_j\tau - \frac{K_{1j}^{\circ}}{b_j} \cos b_j\tau - \frac{K_{2j}}{2}\left[\frac{\cos(b'+b_j)\tau}{b'+b_j} + \frac{\cos(b'-b_j)\tau}{b'-b_j} + \frac{\sin(a'-b_j)\tau}{a'-b_j} - \frac{\sin(a'+b_j)\tau}{a'+b_j} \right] \\ \delta (N_j \sin M_j) &= \frac{K_{1j}^{\circ}}{b_j} \sin b_j\tau + \frac{K_{1j}}{b_j} \cos b_j\tau - \frac{K_{2j}}{2}\left[-\frac{\cos(a'+b_j)\tau}{a'+b_j} - \frac{\cos(a'-b_j)\tau}{a'-b_j} - \frac{\sin(b'+b_j)\tau}{b'+b_j} + \frac{\sin(b'-b_j)\tau}{b'-b_j} \right]\end{aligned} \tag{38}$$

Inserting these values in (37), we find:

$$\begin{aligned}\delta P_i &= \Sigma_j\, \sigma_{ij} \left[\frac{K_{1j}^{\circ}}{b_j} - K_{2j}\left(\frac{b' \cos b'\tau}{b'^2 - b_j^2} + \frac{b_j \sin a'\tau}{a'^2 - b_j^2} \right) \right], \\ \delta Q_i &= \Sigma_j\, \sigma_{ij} \left[\frac{K_{1j}}{b_j} + K_{2j}\left(-\frac{b_j \sin b'\tau}{b'^2 - b_j^2} + \frac{a' \cos a'\tau}{a'^2 - b_j^2} \right) \right].\end{aligned} \tag{39}$$

As appears from the expressions (39) the corrections δP_i and δQ_i do not contain any terms proportional to the time. From (39) we can draw some conclusions as to the different parts of which δP_i and δQ_i consist. It is very probable that the constant part of δP_i is small, because the constants K_{1j} and K_{1j}° are very small and undoubtedly much smaller than b_j ($j = 1, \ldots 4$), which is easily verified by regarding u_j and v_j as ratios of two determinants. It then appears that the determinant of the numerator contains a column of very small terms $(i5)p_{50}'$, and $(i5)q_{50}'$, that are much smaller than the σ_{ij}, while the determinant of the denominator consists only of σ_{ij}, and its value will probably not differ much from the value of the diagonal, which is 1. In the case of $j = 5$, the constants K_{15} and K_{15}° are small, but $\sigma_{i5} \frac{K_{15}}{b_5}$ then contains the denominator b_5 which is also small; the constant σ_{i5} being small, it may be expected that even this last term will be fairly small, except however, in the case when $j = 5$. Finally we thus conclude: the constant parts of δP_i and δQ_i are small, except perhaps that of δP_5 and δQ_5.

As to the periodic terms of δP_i and δQ_i they are for the greater part very small, because of the coefficients b' and a' which are about $\frac{1}{25}$ of b_5 and very much smaller than the other b_j. Exceptions are again formed by δP_5 and δQ_5, as may easily be seen. We may, however, prefer to develop the periodic terms in powers of τ. The argument being $a'\tau$ and $b'\tau$, we can entirely neglect powers higher than the second especially as they are multiplied by relatively small constants.

Thus, developing (39) in terms of τ and neglecting τ^2, we find:

$$\delta P_i = \Sigma_j \sigma_{ij} \left[- \frac{K_{ij}^{\circ}}{b_j} - \frac{K_{2j}\, b'}{b'^2 - b_j^2} - \frac{K_{2j}\, b_j\, a'}{a'^2 - b_j^2}\, \tau \right]$$

$$\delta Q_i = \Sigma_j \sigma_{ij} \left[+ \frac{K_{1j}}{b_j} + \frac{K_{2j}\, a'}{a'^2 - b_j^2} - \frac{K_{2j}\, b_j\, b'}{b'^2 - b_j^2}\, \tau \right]$$

These expressions are more suitable than (39) to give an idea of the value of δP_i and δQ_i and appear to justify our supposition that these quantities are small and, with the possible exception of δP_5, δQ_5, practically negligible.

Souillart has not taken account of these corrections, but Tisserand in his discussion of Souillart's theory *), has given similar expressions and found that in most of the cases the constant parts could be neglected; as to the terms with τ in two cases he found values, that although very small, could not be entirely neglected. It seems to me however that Tisserand's reasoning is not consistent.

As in the numerical part of the integration we shall derive the values of δP_i and δQ_i, it is desirable for the sake of comparison to write down Tisserand's values. He finds:

$$\delta P_1 = + \ 0''.1 \qquad \delta Q_1 = - \ 0''.1$$
$$\delta P_2 = - \ 1.3 \qquad \delta Q_2 = + \ 1.0$$
$$\delta P_3 = - \ 8.1 - 0''.002\ t \qquad \delta Q_3 = + \ 6.0 - 0''.003\ t$$
$$\delta P_4 = - \ 57.4 - 0.013\ t \qquad \delta Q_4 = + \ 42.4 - 0.017\ t$$
$$\delta P_5 = + \ 0.2 \qquad \delta Q_5 = - \ 0.2$$

where t is expressed in Julian years.

(B) *Integration according to method II.*

We now choose the variables:

$$P_1, P_2, P_3, P_4, P_5 = p_s = p_{s0}' + a'\tau + \beta \cos i_0$$
$$Q_1, Q_2, Q_3, Q_4, Q_5 = q_s = q_{s0}' + b'\tau + \alpha \cos i_0$$

The system of differential equations then remains nearly the same as the system (F_3); only the right hand sides become equal to zero in the case of the first four sets, the right hand member of the fifth set becoming a' and b' respectively. We denote this system by (F_3'). If we again neglect the right hand sides of the equations, the integrals are the same as in method I, viz: (33).

In order to integrate the complete system (F_3') regarding N_j and M_j as variables, and using the same notation as in method I, we find:

(40) $$\Sigma_j \sigma_{1j} u_j = 0, \ \ldots\ldots \ \Sigma_j \sigma_{5j} u_j = a'.$$

and:

(41) $$\Sigma_j \sigma_{1j} v_j = 0, \ \ldots\ldots \ \Sigma_j \sigma_{5j} v_j = b'.$$

We then find:

$$u_j = K_j \qquad v_j = K_j^{\circ}$$

where $K_j = \frac{a'}{b'} K_j^{\circ}$. It is again easy to see, that very probably the constants u_j and v_j are small. In each of the equations (40) and (41) a term $\sigma_{ii} u_i$ or $\sigma_{ii} v_i$ occurs. Now $\sigma_{ii} = 1$, while the other σ_{ij} are but

*) Cf. Tisserand IV p. 58.

small, so that the left hand members of the equations will be nearly equal to $\sigma_{ii} u_i$, and as the right hand sides are zero or equal to the small constants a′ and b′, it is probable that u_j and v_j will be small; moreover it may be expected that among the u_j and v_j the largest will be u_5 and v_5, which will but slightly differ from a′ and b′.

Further we have:

$$(42) \qquad \begin{aligned} \delta\,(N_j \cos M_j) &= \frac{K_j}{b_j} \sin b_j\tau - \frac{K_j^\circ}{b_j} \cos b_j\tau \\ \delta\,(N_j \sin M_j) &= \frac{K_j}{b_j} \cos b_j\tau + \frac{K_j^\circ}{b_j} \sin b_j\tau \end{aligned}$$

Substituting (42) in (37) we find:

$$\delta P_i = - \Sigma_j \sigma_{ij} \frac{K_j^\circ}{b_j} \quad , \quad \delta Q_i = + \Sigma_j \sigma_{ij} \frac{K_j}{b_j}$$

whence:

$$\delta Q_i = - \frac{a'}{b'} \delta P_i$$

It is seen at a glance how much simpler the corrections δP_i and δQ_i according to method II are than those of method I.

Having regard to the very small constants K_j and K_j°, we may, with sufficient accuracy simplify the expressions for δP_i and δQ_i, reducing them to:

$$\begin{aligned} \delta P_i &= - \sigma_{ii} \frac{K_i^\circ}{b_i} - \sigma_{i5} \frac{K_5^\circ}{b_5} \\ \delta Q_i &= + \sigma_{ii} \frac{K_i}{b_i} + \sigma_{i5} \frac{K_5}{b_5}. \end{aligned}$$

For the numerical computation these expressions are preferable to the similar ones of method I. The corrections δN_j and δM_j are also simpler than the former. From the equations (36) it follows that

$$\delta N_j = \frac{Kj}{b_j} \sin (b_j\tau + M_j) - \frac{K_j^\circ}{b_j} \cos (b_j\tau + M_j)$$

which shows that δN_j is a very small periodic correction, except in the case when $j = 5$. In the equation for $N_j \frac{dM_j}{d\tau}$ we may regard N_j as a constant and it then appears that:

$$\delta M_j = \frac{K_j}{N_j b_j} \cos (b_j\tau + M_j) + \frac{K_j^\circ}{N_j b_j} \sin (b_j\tau + M_j)$$

As K_j and K_j° contain a′ and b′ as factors and as these quantities are of the order 10^{-9}, while the proper inclinations (see p. 56) are 8×10^{-3} at the utmost, with the exception of δM_5, the correction to M_i is negligible. Moreover it appears that also in this method the integration of the complete equations does not alter b_j, i. e. the motions of the nodes.

Comparison of the results according to the I^st and IInd method.

In the first method of integration we have substitued $\sin a'\tau$ and $\sin b'\tau$ for $a'\tau$ and $b'\tau$. If we make the same substitution in the second method, the corrections δP_i and δQ_i are more complicated than the above. In order to have the same basis in both methods, we give the corrections δP_i and δQ_i in the latter case:

$$\delta P_i = \Sigma_j \sigma_{ij} \left[K_j \left(\frac{a' \sin a'\tau}{a'^2 - b_j^2} + \frac{b'}{a'} \cdot \frac{b_j \cos b'\tau}{b'^2 - b_j^2} \right) \right],$$

$$\delta Q_i = \Sigma_j \sigma_{ij} \left[K_j \left(-\frac{b_j \cos a'\tau}{a'^2 - b_j^2} + \frac{b'}{a'} \cdot \frac{b' \sin b'\tau}{b'^2 - b_j^2} \right) \right]$$

$$j = 1, \ldots, 4, 5.$$

where K_j has the same meaning as above, and the relation $K_j^\circ = \frac{b'}{a'} K_j$ has been used.

The complete values of P_i and Q_i, according to both methods are then, developing in powers of τ and neglecting τ^2, — which is allowable on account of the extreme smallness of the constants a' and b' of the argument, — :

(43) method I

$$\begin{cases} P_i = \Sigma_j \sigma_{ij} N_j \cos(b_j\tau + M_j) + \Sigma_j \sigma_{ij} \left[-\frac{K_{1j}^\circ}{b_j} + \frac{K_{2j}\, b'}{b_j^2} + \frac{K_{2j}\, a'}{b_j} \tau \right] \\ Q_i = \Sigma_j \sigma_{ij} N_j \sin(b_j\tau + M_j) + \Sigma_j \sigma_{ij} \left[-\frac{K_{1j}}{b_j} - \frac{K_{2j}\, a'}{b_j^2} + \frac{K_{2j}\, b'}{b_j} \tau \right] \end{cases}$$

$$j = 1, \ldots 4, 5,$$

but for $j = 5$ the coefficient of τ is more accurately given by the expressions:

$$-\frac{K_{2j}\, a'\, b_j}{a'^2 - b_j^2} \text{ and } -\frac{K_{2j}\, b'\, b_j}{b'^2 - b_j^2} \text{ respectively.}$$

(44) method II

$$\begin{cases} P_i = \Sigma_j \sigma_{ij} N_j \cos(b_j\tau + M_j) + \Sigma_j \sigma_{ij} \left[-\frac{K_j a'^2}{b_j^2} \tau - \frac{b'}{a'} \frac{K_j}{b_j} \right] \\ Q_i = \Sigma_j \sigma_{ij} N_j \sin(b_j\tau + M_j) + \Sigma_j \sigma_{ij} \left[-\frac{K_j b'^3}{a' b_j^2} \tau + \frac{K_j}{b_j} \right] \end{cases}$$

$$j = 1, \ldots 4, 5,$$

whereas for $j = 5$ it is better to take for the constant parts:

$$\frac{b_j b'}{a'(b'^2 - b_j^2)} K_j \text{ and } -\frac{b_j}{a'^2 - b_j^2} K_j$$

It may be expected at first sight that the values of P_i and Q_i as derived from both methods of integration must agree within the limits of accuracy of the formulae:

$$p_s = p_{s0}' + a'\tau + \beta \cos i_0$$
$$q_s = q_{s0}' + b'\tau + \alpha \cos i_0 ,$$

where p_{s0}' and q_{s0}' are trustworthy to 6, and a' and b' to 12 decimal places.

Consequently the constant parts can only differ at the utmost in the sixth, the coefficient of τ only in the twelfth decimal place. On examining the expressions (43) and (44), we find that it is very probable that the coefficients of τ in (44) are much smaller than those in (43). This conclusion, however, is rather vague. Therefore with the values of σ_{ij}, found in the numerical integration, we have computed the constants K_{1j}, K_{1j}°, K_{2j}, K_j, K_j° and then δP_i and δQ_i.

We find, the numbers in square brackets denoting logarithms:

$K_{11} = -[2{\cdot}122\,592 - 10]$	$K_{11}^\circ = [2{\cdot}124\,404 - 10]$	$K_{21} = [3{\cdot}533\,930 - 10]$	$K_1 = -[5{\cdot}873\,258 - 10]\, a'$
$K_{12} = -[2{\cdot}247\,337 - 10]$	$K_{12}^\circ = [2{\cdot}259\,189 - 10]$	$K_{22} = [3{\cdot}668\,715 - 10]$	$K_2 = -[5{\cdot}350\,325 - 10]\, a'$
$K_{13} = -[2{\cdot}717\,179 - 10]$	$K_{13}^\circ = [2{\cdot}728\,991 - 10]$	$K_{23} = [4{\cdot}138\,517 - 10]$	$K_3 = -[8{\cdot}006\,371 - 10]\, a'$
$K_{14} = -[3{\cdot}333\,756 - 10]$	$K_{14}^\circ = [3{\cdot}345\,568 - 10]$	$K_{24} = [4{\cdot}755\,094 - 10]$	$K_4 = -[9{\cdot}195\,724 - 10]\, a'$
$K_{15} = [1{\cdot}130\,257 - 10]$	$K_{15}^\circ = -[1{\cdot}142\,069 - 10]$	$K_{25} = -[2{\cdot}551\,595 - 10]$	$K_5 = [10{\cdot}000\,182 - 10]\, a'$

$$\log a' = -[1{\cdot}287\,130 - 10] \qquad \log b' = [1{\cdot}149\,835 - 10]$$

The values of δP_i and δQ_i then are, for the two methods:

method I

$$\delta P_1 = -\ 0{\cdot}0000\ 18866 \qquad \delta Q_1 = -\ 0{\cdot}0000\ 25880$$
$$\delta P_2 = -\ 0{\cdot}0002\ 02205 \qquad \delta Q_2 = -\ 0{\cdot}0002\ 77388$$
$$\delta P_3 = -\ 0{\cdot}0011\ 40762 - 0{\cdot}0000\ 0000\ 000063\ \tau \qquad \delta Q_3 = -\ 0{\cdot}0015\ 64911 - 0{\cdot}0000\ 0000\ 000087\ \tau$$
$$\delta P_4 = -\ 0{\cdot}0063\ 69151 - 0{\cdot}0000\ 0000\ 000350\ \tau \qquad \delta Q_4 = -\ 0{\cdot}0087\ 37284 - 0{\cdot}0000\ 0000\ 000480\ \tau$$
$$\delta P_5 = -\ 0{\cdot}0011\ 80690 + 0{\cdot}0000\ 0000\ 195837\ \tau \qquad \delta Q_5 = -\ 0{\cdot}0016\ 19686 - 0{\cdot}0000\ 0000\ 268651\ \tau$$

whence it appears that if P_i and Q_i must be accurate to 6 decimal places during a century, the correction proportional to the time cannot be neglected.

method II

$$\delta P_1 = -\ 0{\cdot}0000\ 1904 \qquad \delta Q_1 = -\ 0{\cdot}0000\ 2611$$
$$\delta P_2 = -\ 0{\cdot}0002\ 0888 \qquad \delta Q_2 = -\ 0{\cdot}0002\ 8654$$
$$\delta P_3 = -\ 0{\cdot}0011\ 3286 \qquad \delta Q_3 = -\ 0{\cdot}0015\ 5408$$
$$\delta P_4 = -\ 0{\cdot}0063\ 1793 \qquad \delta Q_4 = -\ 0{\cdot}0086\ 6703$$
$$\delta P_5 = -\ 0{\cdot}0397\ 9136 + 0{\cdot}0000\ 0000\ 00058\ \tau \qquad \delta Q_5 = -\ 0{\cdot}0545\ 8636 - 0{\cdot}0000\ 0000\ 00022\ \tau$$

The corrections proportional to the time thus are, even in a century, entirely negligible. It should be noted that the corrections δP_i and δQ_i according to the second method are rigorously constant, as was found above. The small terms with τ have been introduced by the development of $\sin a'\tau$ and $\sin b'\tau$, which were introduced instead of a' and b' for the sake of the comparison of the two methods.

The corrections δP_i and δQ_i according to the two methods do not agree, and the discrepancies are too large to be attributed to the uncertainty of the numerical computations. It therefore seems as if the methods are not equivalent. This however is not the case, as can be shown by the following reasoning:

The integrals according to the first method are:

$$P_i = \Sigma_j \sigma_{ij} N_j \cos(b_j\tau + M_j) + \delta P_i$$
$$Q_i = \Sigma_j \sigma_{ij} N_j \sin(b_j\tau + M_j) + \delta Q_i$$

where: $P_5 = \beta \cos i_o$, $Q_5 = \varkappa \cos i_o$, and δP_i and δQ_i can be written in the form:

$$\delta P_i = c_i + c_{io}\tau$$
$$\delta Q_i = c'_i + c_{io}'\tau$$

c_i, c_i', c_{io}, c_{io}' are constants.

According to the second method of integration we have:

$$P_i = \Sigma_j \sigma_{ij} N_j \cos(b_j\tau + M_j) + \delta P_i$$
$$Q_i = \Sigma_j \sigma_{ij} N_j \sin(b_j\tau + M_j) + \delta Q_i$$

where:

$$P_5 = p_{so}' + \sin a'\tau + \beta \cos i_o \quad , \quad Q_5 = q_{so}' + \sin b'\tau + \varkappa \cos i_o ,$$
$$\delta P_i = \varkappa_i + \varkappa_{io}\tau$$
$$\delta Q_i = \varkappa_i' + \varkappa_{io}'\tau$$

$\varkappa_i$, $\varkappa_i'$, $\varkappa_{io}$, $\varkappa_{io}'$ are constants.

Now the form of the integrals in both cases is the same, but the integration-constants N_j and M_j are different, because in order to find them, we must substitute $\tau = 0$, and then find:

(a)
$$\Sigma_j \sigma_{ij} N_j \cos M_j = P_i(\tau=0) = c_i$$
$$\Sigma_j \sigma_{ij} N_j \sin M_j = Q_i(\tau=0) = c_i'$$

where the right hand members contain in the case $i = 5$:

$$\beta \cos i_o (\tau = 0) \text{ and } \alpha \cos i_o (\tau = 0).$$

In the second case we have:

$$(b) \qquad \begin{aligned} \Sigma_j \sigma_{ij} N_j \cos M_j &= P_i (\tau = 0) - \varkappa_i \\ \Sigma_j \sigma_{ij} N_j \sin M_j &= Q_i (\tau = 0) - \varkappa_i' \end{aligned}$$

while the right hand sides of the equations (b) are, for $i = 5$, :

$$p_{so}' + \beta \cos i_o (\tau = 0) \quad , \quad q_{so}' + \alpha \cos i_o (\tau = 0).$$

Evidently the systems of equations (a) and (b) are not identical. Let the solutions $N_j \cos M_j$ and $N_j \sin M_j$ of the system (a) be respectively A_j and $- B_j$, that of the set (b) similarly: A_j' and $- B_j'$.

We then have in the case of Method I:

$$\begin{aligned} P_i^{\text{I}} &= \Sigma_j \sigma_{ij} (A_j \cos b_j \tau + B_j \sin b_j \tau) \\ Q_i^{\text{I}} &= \Sigma_j \sigma_{ij} (A_j \sin b_j \tau - B_j \cos b_j \tau). \end{aligned}$$

The second method, on the other hand, gives:

$$(45) \qquad \begin{aligned} P_i^{\text{II}} = \Sigma_j \sigma_{ij} (A_j' \cos b_j \tau + B_j' \sin b_j \tau) &= \Sigma_j \sigma_{ij} (A_j \cos b_j \tau + B_j \sin b_j \tau) + \Sigma_j \sigma_{ij} (A_j' - A_j) \cos b_j \tau \\ &\quad + \Sigma_j \sigma_{ij} (B_j' - B_j) \sin b_j \tau \\ Q_i^{\text{II}} = \Sigma_j \sigma_{ij} (A_j' \sin b_j \tau - B_j' \cos b_j \tau) &= \Sigma_j \sigma_{ij} (A_j \sin b_j \tau) - B_j \cos b_j \tau) + \Sigma_j \sigma_{ij} (A_j' - A_j) \sin b_j \tau \\ &\quad - \Sigma_j \sigma_{ij} (B_j' - B_j) \cos b_j \tau \end{aligned}$$

or, developing the terms with the coefficients $A_j' - A_j$ and $B_j' - B_j$ in powers of τ and neglecting τ^2, we find:

$$\begin{aligned} P_i^{\text{II}} &= P_i^{\text{I}} + \Sigma_j \sigma_{ij} (A_j' - A_j) + \tau \Sigma_j \sigma_{ij} (B_j' - B_j) b_j = P_i^{\text{I}} + \varepsilon_i + \varepsilon_{io} \tau \\ Q_i^{\text{II}} &= Q_i^{\text{I}} - \Sigma_j \sigma_{ij} (B_j' - B_j) + \tau \Sigma_j \sigma_{ij} (A_j' - A_j) b_j = Q_j^{\text{I}} + \varepsilon_i' + \varepsilon_{io}' \tau \end{aligned}$$

These expressions show that, in order that the second method may be comparable with the first, not only the corrections $\varkappa_i + \varkappa_{io} \tau$ and $\varkappa_i' + \varkappa_{io}' \tau$, but also $- \varepsilon_i - \varepsilon_{io} \tau$ and $- \varepsilon_i' - \varepsilon_{io}' \tau$ must be applied to the integrals P_i and Q_i of method II.

It is easily verified that the corrections ε_i, etc. actually render account of the differences of the δP_i and δQ_i in both methods, for subtracting the corresponding equations (a) and (b), we find:

$$(c) \qquad \begin{aligned} \Sigma_j \sigma_{ij} (A_j - A_j') &= - c_i + \varkappa_i = \mu_i \\ \Sigma_j \sigma_{ij} (B_j' - B_j) &= - c_i' + \varkappa_i' = \mu_i'. \end{aligned}$$

The solutions of the systems (c) then are:

$$\begin{aligned} A_j - A_j' &= \Sigma_j \varkappa_{ij} \mu_i \\ B_j' - B_j &= \Sigma_j \varkappa_{ij} \mu_i'. \end{aligned}$$

Because of the values of the σ_{ij} ($\sigma_{ii} = 1$), we find that $\varkappa_{ii}$ is nearly 1, and that the other $\varkappa_{ij}$ are small fractions, whence we have approximately:

$$(d) \qquad \begin{aligned} A_j - A_j' &= \mu_i \\ B_j' - B_j &= \mu_i' \end{aligned}$$

From the equations (*c*) we conclude that ε_i and ε_i' wholly explain the differences between δP_i and δQ_i in both methods. As to the corrections ε_{io} and ε_{io}', having regard to the numerical values of the $A_j - A_j'$ and $B_j' - B_j$ and the values of the b_j (see p. 61), it appears at first sight that the $\varepsilon_{io}\tau$ and $\varepsilon_{io}'\tau$ will be larger than the small corrections $c_{io}\tau$ and $c_{io}'\tau$, etc.; but in this case also a more thorough investigation will prove that the ε_{io} and ε_{io}' will be sufficiently near the differences of c_{io}, c_{io}' and $\varkappa_{io}$, $\varkappa_{io}'$, since the b_j, $A_j - A_j'$, $B_j' - B_j$ and σ_{ij} are of very different amounts and hence the terms $\sigma_{ij}(B_j' - B_j)\,b_j$, etc. may be almost equal and cancel one another, so as to make $\Sigma_j \sigma_{ij}(B_j' - B_j)\,b_j$ and $\Sigma_j \sigma_{ij}(A_j' - A_j)\,b_j$ very small.

The difference between the methods I and II can also be interpreted geometrically. As appears from the formulae (45) the co-ordinates P_i and Q_i in both methods have not the same meaning; they refer to a different system of axes. The pole of the plane of (x, y) in the second case describes a curve about that of the plane of (x, y) in the first case and this periodic motion is reflected as the periodic corrections that must be applied to the co-ordinates of case II in order to make the results agree.

In the preceding it has been shown that both methods give the same results; in the following the second method — being the simpler of the two — has been adopted. From the expressions δP_i and δQ_i it may be concluded that the right-hand members of the equations (F_3), (F_3') cause an alteration of the initial values of P_i and Q_i; these alterations are negligible for the first satellite, in the case of the fourth satellite and the Sun they become considerable and cannot be neglected.

Comparing the values of δP_i and δQ_i according to the last method with those of Tisserand we find that δP_i and δQ_i in both cases increase with the mean distance a_i, except δP_5 and δQ_5, where the values of Tisserand must be rejected. Our corrections are larger, but it must be remembered that by Tisserand's reasoning the influence of the fifth root b_5 on δP_i and δQ_i has been omitted, which is the reason, why his corrections are too small.

The equations according to method II are (F_3'). The integrals of (F_3') then are:

$$P_i = \Sigma_j \sigma_{ij} N_j \cos(b_j\tau + M_j) + \delta P_i$$
$$Q_i = \Sigma_j \sigma_{ij} N_j \sin(b_j\tau + M_j) + \delta Q_i$$

The values of δP_i and δQ_i are given on p. 45, viz:

$$\delta P_i = -\Sigma_j \sigma_{ij} \frac{K_j^\circ}{b_j} \quad , \quad \delta Q_i = \Sigma_j \sigma_{ij} \frac{K_j}{b_j}$$

We find the following numerical values:

$\delta P_1 = -$ 0·0000 1904	$\delta Q_1 = -$ 0·0000 2611
$\delta P_2 = -$ 0·0002 0888	$\delta Q_2 = -$ 0·0002 8654
$\delta P_3 = -$ 0·0011 3286	$\delta Q_3 = -$ 0·0015 5408
$\delta P_4 = -$ 0·0063 1793	$\delta Q_4 = -$ 0·0086 6703
$\delta P_5 = -$ 0·0397 9136	$\delta Q_5 = -$ 0·0545 8636

In order to compute the influence of the part R^{IV} of the perturbative function on the b_j (see Chapter VI) we need the numerical values of the N_j. They can be derived from the initial values of P_i and Q_i.

Souillart's theory furnishes values of P_i and Q_i, that involve secular inequalities but not the corrections δP_i, δQ_i, which have just been derived. We therefore are obliged, in order to derive approximate values of the N_j, to use the values of P_i and Q_i derived by prof. de Sitter from Souillart's

theory (*"Elements and Masses"* p. 723). The values there given refer to the mean, instead of to the true equator of Jupiter, and depend on a set of masses and elements, differing from ours, but since the values of N_j derived from them are only approximations, this is of no importance.

It should be noticed, that the orbital planes of the satellites are there determined by (Cf. loc. cit. p. 722):

$$p_i = i_i \sin(\theta' - \Omega_i) \quad , \quad q_i = i_i \cos(\theta' - \Omega_i)$$

and Jupiter's true equator by means of (Cf loc. cit. p. 723):

$$\alpha_1 = \omega_1 \sin(\theta' - \psi_1) \quad , \quad \beta_1 = \omega_1 \cos(\theta' - \psi_1)$$

where Ω_i is the ascending node of a satellite's orbit and ψ_1 is the ascending node of Jupiter's true equator.

We have put, neglecting σ_i:

$$P_i = \sqrt{G_i} \sin i_i \cos \Omega_i \quad , \quad Q_i = -\sqrt{G_i} \sin i_i \sin \Omega_i$$
$$\alpha = \sin \psi \sin \pi \quad , \quad \beta = \cos \psi \sin \pi .$$

Hence we find, omitting i_i^3:

$$P_i = \sqrt{G_i}\,(q_i \cos\theta' + p_i \sin\theta') \quad , \quad Q_i = -\sqrt{G_i}\,(q_i \sin\theta' - p_i \cos\theta')$$
$$\beta = -[\beta_1 \cos\theta' + \alpha_1 \sin\theta'] \quad , \quad \alpha = \beta_1 \sin\theta' - \alpha_1 \cos\theta'$$

since: $$\psi_1 = 180^\circ - \psi \text{ (see p. 7)}.$$

Employing the formulae of transformation:

$$P_i = P_{io} + \beta \quad , \quad Q_i = Q_{io} + \alpha$$

— where P_i and Q_i refer to the true, P_{io} and Q_{io} to the mean equator —, P_i, Q_i, α and β can be computed. The formulae of de Sitter are:

$$\begin{aligned}
p_{1o} &= +\ 0^\circ02720 \sin\Gamma_1 + 0^\circ00951 \sin\Gamma_2 + 0^\circ00103 \sin\Gamma_3 - 0^\circ00046 \sin\Gamma_4 \\
p_{2o} &= -\ \cdot00052 + \cdot46830 + \cdot02734 + \cdot00464 \\
p_{3o} &= -\ \cdot00003 - \cdot01625 + \cdot18390 + \cdot03051 \\
p_{4o} &= \cdot00000 - \cdot00047 - \cdot03259 + \cdot25360
\end{aligned}$$

The expressions of q_{io} are the same, with $\cos\Gamma_i$ instead of $\sin\Gamma_i$, and the values of q_{io} must be increased by the small constants: 0°00174 , 0°01792 , 0°08502 , 0°42851 respectively.

Further we have:

$$\alpha_1 = \Sigma_j \sigma_{oj}\gamma_j \sin\Gamma_j \quad , \quad \beta_1 = \Sigma_j \sigma_{oj}\gamma_j \cos\Gamma_j \quad , \quad j = 1, \ldots 4.$$

where:

$$\begin{aligned}
\sigma_{o1}\gamma_1 &= -\ 0^\circ00003 & \Gamma_1 &= 75^\circ6 + 0^\circ136\,18\ t \\
\sigma_{o2}\gamma_2 &= -\ 0\cdot00044 & \Gamma_2 &= 202\cdot64 + 0\cdot032\,373\ t \\
\sigma_{o3}\gamma_3 &= -\ 0\cdot00081 & \Gamma_3 &= 176.09 + 0\cdot006\,892\ t \\
\sigma_{o4}\gamma_4 &= -\ 0\cdot00092 & \Gamma_4 &= 123\cdot84 + 0\cdot001\,810\ t
\end{aligned}$$

where t is counted in mean solar days from 1900 Jan 0·0 M. T. Greenwich.

The values of P_i $(\tau = 0)$, Q_i $(\tau = 0)$ then are:

$$\begin{aligned}
P_1 &= +\ 0\cdot0002\,56 & Q_1 &= -\ 0\cdot0002\,25 \\
P_2 &= +\ 0\cdot0068\,88 & Q_2 &= +\ 0\cdot0028\,59 \\
P_3 &= +\ 0\cdot0018\,28 & Q_3 &= +\ 0\cdot0006\,60 \\
P_4 &= -\ 0\cdot0016\,61 & Q_4 &= -\ 0\cdot0075\,25 \\
P_5 &= +\ 0\cdot0000\,2974\ (=\beta\cos i_o) & Q_5 &= -\ 0\cdot0000\,1244\ (=\alpha\cos i_o)
\end{aligned}$$

The values of $A_i = N_i \cos M_i$, $B_i = -N_i \sin M_i$ derived from equations (b), taking into account the corrections x_i, x_i', and using the preceding values of P_i and Q_i, are:

$$\begin{aligned} A_1 &= +\ 0{\cdot}000\,1090 & B_1' &= +\ 0{\cdot}000\,2822 \\ A_2 &= +\ 0{\cdot}006\,5314 & B_2 &= -\ 0{\cdot}003\,5671 \\ A_3 &= +\ 0{\cdot}002\,6239 & B_3 &= -\ 0{\cdot}001\,9738 \\ A_4 &= +\ 0{\cdot}005\,0611 & B_4 &= +\ 0{\cdot}001\,0325 \\ A_5 &= +\ 0{\cdot}000\,8431 & B_5 &= -\ 0{\cdot}016\,6640 \end{aligned}$$

and from these we find:

$N_1 = 0{\cdot}0003\,025$, $N_2 = 0{\cdot}0074\,420$, $N_3 = 0{\cdot}0032\,834$, $N_4 = 0{\cdot}0051\,654$, $N_5 = 0{\cdot}016\,685$.

Analytical integration of the equation for σ_i .

The differential equation for σ_i is:

$$\frac{d\sigma_i}{d\tau} = -\frac{1}{2\,G_i}\left(p_{io}\frac{\partial R_i}{\partial p_{io}} + q_{io}\frac{\partial R_i}{\partial q_{io}}\right)$$

Neglecting quantities of the order of p_i^3 and q_i^3, on the right hand side p_{io} and q_{io} can be replaced by p_i and q_i. Substituting the values of $\frac{\partial R_i}{\partial p_i}$ and $\frac{\partial R_i}{\partial q_i}$, the equation assumes the form:

$$(46)\qquad \frac{d\sigma_i}{d\tau} = -\frac{1}{2\,G_i}\left[\,2\,(A_i + \Sigma_j B_{ij})\,p_i^2 - 2\,\Sigma_j B_{ij} p_i p_j + 2\,(A_i + \Sigma_j B_{ij})\,q_i^2 - 2\,\Sigma_j B_{ij} q_i q_j\,\right]$$

where $i \neq j$. This equation can be integrated by substituting on the right hand side the integrals:

$$\begin{aligned} P_i &= \Sigma_j \sigma_{ij}\ N_j \cos(b_j\tau + M_j) \\ Q_i &= \Sigma_j \sigma_{ij}\ N_j \sin(b_j\tau + M_j)\,. \end{aligned}$$

It then appears that the right hand member of (46) contains constant and periodic parts. Neglecting the periodic parts on account of their small coefficients as a quite sufficient approximation, the equation becomes:

$$\frac{d\sigma_i}{d\tau} = -\frac{1}{G_i^2}\,(A_i + \Sigma_j B_{ij})\ \Sigma_\lambda \sigma_{i\lambda}^2 N_\lambda^2 - \frac{1}{G_i}\left[\Sigma_j \frac{B_{ij}}{G_j}\,\Sigma_\mu \sigma_{i\mu}\sigma_{j\mu} N_\mu^2\right] = [A_i]\,.$$

Integrating we thus have:

$$(47)\qquad \sigma_i = [A_i]\,\tau + C_i$$

This small correction is to be applied to the values found for the nodes. Unless extreme accuracy is required, it can be omitted. The expression of $[A_i]$ is rather intricate, but since the greater part of σ_{ij} is small compared with $\sigma_{ii} = 1$, the expression of $[A_i]$ can be reduced to:

$$[A_i] = -\frac{1}{G_i^2}\,(A_i + \Sigma_j B_{ij})\ N_i^2$$

showing that $[A_i]$ is proportional to the square of the proper inclination of satellite i.

Numerical integration of the differential equations.

The coefficients $[i]$, (ij), etc, of the differential equations (F_3') contain the constants $d_{ij}^{(1)}$, which are functions of the coefficients $b^{(i)}$ of Laplace. In the first place these $b^{(i)}$ must be determined. Souillart in his second memoir p. 11 has given the values of $b^{(i)}$. They have been computed from Runkle's tables and checked by computing them by at least one other method, so that the data of Souillart can be taken as absolutely trustworthy. We cannot, however, use them as they stand, since the values of α, used by Souillart $\left(\alpha = \frac{a_i}{a'}\right)$ differ from ours, but the differences are very small. Consequently the corrections to be applied to the $b^{(i)}$ of Souillart were small.

Souillart, however, has not computed the $b^{(i)}$ further than $i = 12$ at the utmost. This was quite sufficient for the purpose he had set himself. But with the degree of accuracy necessary for the computation of the $[i]$, etc., the $d_{ij}^{(1)}$ must be known to 6 decimal places, and therefore several $b^{(i)}$ must be computed. This has been done by means of the formulae given by *Leverrier* in the "*Annales de l'Observatoire de Paris*", Tome II, add. 1re, N° II. (The formulae there go as far as $b^{(15)}$, but the remaining $b^{(i)}$ have been derived from these by means of the formulae of p. 16 loc. cit.)

For the computation of the $d_{ij}^{(1)}$ we only need the $b^{(i)}$ with even i; we have, however, also computed the $b^{(i)}$ for i uneven. Moreover the directly computed $b^{(i)}$ have been compared with the $b^{(i)}$ computed by adding corrections to Souillart's values; it appeared that some of the coefficients $b^{(i)}$, as determined in both ways did not agree, though the differences were very small. In these cases we have adopted the values obtained by direct computation.

The corrections to the $b^{(i)}$ of Souillart have been computed by the formula:

$$\Delta b^{(i)} = \frac{h}{\alpha}\,\alpha\,\frac{db^{(i)}}{d\alpha} + \frac{h^2}{2\alpha^2}\,\alpha^2\,\frac{d^2 b^{(i)}}{d\alpha^2} + \ldots\ldots$$

where $h = [\alpha] - \alpha$, α being the value of Souillart, and $[\alpha]$ ours. The numerical values of $\alpha^n \frac{d^n b^{(i)}}{d\alpha^n}$ are the $b_n^{(i)}$ of Souillart.

Satellites I and II.

$\log [\alpha] = 9{\cdot}7983165 - 10 \qquad [\alpha] = 0{\cdot}6285162$

i	$b^{(i)}$	i	$b^{(i)}$	i	$b^{(i)}$
2	0·3632 1087	14	0·0005 7084	24	0·0000 0422
4	0·1065 0823	15	0·0003 4705	25	0·0000 0260
6	0·0349 9112	16	0·0002 1143	26	0·0000 0160
7	0·0204 7405	17	0·0001 2904	27	0·0000 0099
8	0·0120 8847	18	0·0000 7888	28	0·0000 0060
9	0·0071 8707	19	0·0000 4829	29	0·0000 0037
10	0·0042 9720	20	0·0000 2960	30	0·0000 0022
11	0·0025 8105	21	0·0000 1817	31	0·0000 0014
12	0·0015 5614	22	0·0000 1116	32	0·0000 0007
13	0·0009 4122	23	0·0000 0730		

Souillart's value of α is 0·6285 15074 and the values $b^{(i)}$ derived from Souillart's table are:

$b^{(3)} = 0{\cdot}3632\,108$, $b^{(4)} = 0{\cdot}1065\,082$, $b^{(6)} = 0{\cdot}0349\,9114$, $b^{(8)} = 0{\cdot}0120\,8827$

Satellites I and III.

$\log [x] = 9{\cdot}5955\,236 - 10 \qquad [x] = 0{\cdot}3940\,248$

i	$b^{(i)}$	i	$b^{(i)}$	i	$b^{(i)}$
2	0·1248 4362	10	0·0000 34444	15	0·0000 00268
4	0·0142 1485	11	0·0000 1296	16	0·0000 00102
6	0·0018 2529	12	0·0000 04895	17	0·0000 00039
8	0·0002 4702	13	0·0000 01854	18	0·0000 00014
9	0·0000 91978	14	0·0000 00705		

Souillart's value of x is 0·3940 1908 75 and the values of $b^{(i)}$:

$b^{(2)} = 0{\cdot}1248\,438, \quad b^{(4)} = 0{\cdot}0142\,1482, \quad b^{(6)} = 0{\cdot}0018\,251$

Satellites I and IV.

$\log [x] = 9{\cdot}3503\,069 - 10 \qquad [x] = 0{\cdot}2240\,304$

i	$b^{(i)}$	i	$b^{(i)}$
2	0·0384 5634	8	0·0000 02553
4	0·0014 0981	9	0·0000 00540
5	0·0002 8438	10	0·0000 00115
6	0·0000 58418	11	0·0000 00025
7	0·0000 12155	12	0·0000 00005

Souillart's value of x is 0·2240 19672 and the value $b^{(2)}$ derived from Souillart exactly agrees with ours.

Satellite I and Sun.

$[x] = 0{\cdot}0005\,4188 \qquad \log [x] = 6{\cdot}733903 - 10$

$[x]^4$ can be neglected, whence also $b_{is}^{(4)}$, etc; thus in this case we have $d_{is}^{(1)} = 2\, b_{is}^{(2)}$, where:

$b_{is}^{(2)} = [9{\cdot}8750\,61 - 10]\, x^2$.

Satellites II and III.

$\log [x] = 9{\cdot}7972\,071 - 10 \qquad [x] = 0{\cdot}6269\,128$

i	$b^{(i)}$	i	$b^{(i)}$	i	$b^{(i)}$
2	0·3608 8801	14	0·0005 49938	24	0·0000 03966
4	0·1052 7458	15	0·0003 33419	25	0·0000 02330
6	0·0344 0756	16	0·0002 02643	26	0·0000 01498
7	0·0200 8089	17	0·0001 23299	27	0·0000 00736
8	0·0118 2588	18	0·0000 75233	28	0·0000 00563
9	0·0070 1275	19	0·0000 45868	29	0·0000 00280
10	0·0041 8245	20	0·0000 28084	30	0·0000 00212
11	0·0025 3293	21	0·0000 17120	31	0·0000 00107
12	0·0015 0682	22	0·0000 10533	32	0·0000 00070
13	0·0007 3607	23	0·0000 06376		

Souillart's x is 0·6269 04754 and the values $b^{(i)}$ derived from his tables are:

$b^{(2)} = 0{\cdot}3608\,878 \quad b^{(4)} = 0{\cdot}1052\,746 \quad b^{(6)} = 0{\cdot}0344\,075$

Satellites II and IV

$\log [\alpha] = 9\cdot5519\,9037 - 10 \qquad [\alpha] = 0\cdot3564\,428$

i	$b^{(i)}$	i	$b^{(i)}$	i	$b^{(i)}$
2	0·1007 9852	8	0·0001 09100	13	0·0000 00496
4	0·0093 81375	9	0·0000 36742	14	0·0000 00170
5	0·0030 13050	10	0·0000 12445	15	0·0000 00059
6	0·0009 85306	11	0·0000 04235	16	0·0000 00020
7	0·0003 26325	12	0·0000 01447	17	0·0000 00007

Souillart's α is 0·3564 26889. The only $b^{(i)}$ that could be derived from Souillart's table was $b^{(2)}$ for which we find: $b^{(2)} = 0.1007\,982$.

Satellite II and Sun

$\log [\alpha] = 6\cdot935586 - 10 \qquad [\alpha] = 0\cdot0008\,62156$

$[\alpha]^4$ can be neglected we then have:

$$[b]^{(2)} = [9\cdot875061 - 10]x^2 = 0\cdot0000\,0055\,7485$$

Satellites III and IV

$\log [\alpha] = 9\cdot7547\,8327 - 10 \qquad [\alpha] = 0\cdot5685\,692$

i	$b^{(i)}$	i	$b^{(i)}$	i	$b^{(i)}$
2	0·2843 1750	15	0·0000 73144	24	0·0000 00360
4	0·0679 4400	16	0·0000 40304	25	0·0000 00202
6	0·0182 3132	17	0·0000 22250	26	0·0000 00111
8	0·0051 48424	18	0·0000 12303	27	0·0000 00062
10	0·0014 9663	19	0·0000 06817	28	0·0000 00034
11	0·0008 14939	20	0·0000 03777	29	0·0000 000196
12	0·0004 43273	21	0·0000 02097	30	0·0000 000095
13	0·0002 42487	22	0·0000 01165	31	0·0000 000055
14	0·0001 33018	23	0·0000 00648		

Souillart's α is 0·5685 50305 and the values $b^{(i)}$ are:

$$b^{(2)} = 0\cdot2843\,177, \qquad b^{(4)} = 0\cdot0679\,4397, \qquad b^{(6)} = 0\cdot0182\,3118$$

Satellite III and Sun.

$\log [\alpha] = 7\cdot138\,379 - 10 \qquad b^{(2)} = 0\cdot0000\,0141\,847$

Satellite IV and Sun.

$\log [\alpha] = 7\cdot383\,596 - 10 \qquad [\alpha] = 0\cdot0024\,1878$

$b^{(2)} = [9\cdot875\,061 - 10]\,x^2 + [9\cdot494\,850 - 10]\,x^4 + \ldots = 0\cdot000\,004\,387\,86$

We now find for $d_{ij}^{(1)} = 2\,b^{(2)} + 4\,b^{(4)} + 6\,b^{(6)} + \ldots\ldots + 2n\,b^{(2n)} + \ldots\ldots$ the values given in the following tables.

$d_{ij}^{(1)}$

i \ j	1	2	3	4	s
1	—	1·534 556	0·3198 896	0·082 9241	0·0000 0044 0450
2	1·534 556	—	1·5170 83	0·246 0517	0·0000 0111 497
3	0·319 8896	1·517 083	—	1·014 1144	0·0000 0283 694
4	0·082 9241	0·246 0517	1·0141 144	—	0·0000 0877 572
s	0·0000 0044 0450	0·0000 0111 497	0·0000 0283 694	0·0000 0877 572	—

The definition of $c_{ij}^{(2)}$ is: $c_{ij}^{(2)} = 4\cdot3\,b^{(4)} + 2\cdot6\cdot4\,b^{(6)} + 3\cdot8\cdot5\,b^{(8)} + \ldots\ldots$ We find:

$c_{ij}^{(2)}$

i \ j	1	2	3	4	5
1	—	6·9183	0·29876	0.0200	negligible.
2	6·9183	—	6·7614	0.17668	negligible.
3	0·29876	6·7614	—	3·0139	0·000 000 000 0235
4	0·0200	0·17668	3·0139	—	0·000 000 000 225
s	negligible	negligible	0·000 000 000 0235	0·000 000 000 225	—

Tabular values for the constants A_i, B_{ij}, C_i.

The definition of these quantities is:

$$A_i = A_i^{\mathrm{I}} + A_i^{\mathrm{II}} \qquad A_i^{\mathrm{I}} = -\tfrac{1}{2}\frac{J_i}{a_i}\mu_i\left(1+\tfrac{3}{2}e_i^2\right) \qquad A_i^{\mathrm{II}} = -\tfrac{1}{2}\frac{K_i}{a_i}\mu_i(1+5e_i^2)$$

$$C_i = C_i^{\mathrm{I}} + C_i^{\mathrm{II}} \qquad C_i^{\mathrm{I}} = \frac{m_i}{\mu_i}A_i^{\mathrm{I}} \qquad C_i^{\mathrm{II}} = \frac{m_i}{\mu_i}A_i^{\mathrm{II}}$$

$$4\,B_{ij} = -\frac{m_j}{a'}\,d_{ij}^{(1)} \qquad \mu_i = 1 + \frac{m_i}{m_\circ} \qquad J_i = \frac{Jb^2}{a_i^2} \qquad K_i = \frac{Kb^4}{a_i^4}$$

i	$\log \frac{J_i}{a_i}$	$\log \frac{K_i}{a_i}$	$\log \frac{\mu_i}{2}$	$\log\left(1+\frac{3}{2}e_i^2\right)$
1	7·198 9173 — 10	4·651 730 — 10	9·698 9870 — 10	0·000 0106
2	6·593 8647 — 10	3·643 315 — 10	6·698 9808 — 10	0·000 0573
3	5·985 490 — 10	2·629 350 — 10	9·699 006 — 10	0·000 002
4	5·249 839 — 10	1·403 265 — 10	9·698 989 — 10	0·000 036
s	7·400 627 — 20	8·321 245 — 30	2·719 580	0·001 531

i	$\log(1+5e_i^2)$	$\log A_i^{\mathrm{I}}$	$\log A_i^{\mathrm{II}}$	$\log 2\,C_i^{\mathrm{I}}$	$\log 2\,C_i^{\mathrm{II}}$
1	0·000 035	6·897·9149 — 10_n	4·350 752 — 10_n	2·789 994 — 10_n	0·242 830 — 10_n
2	0·000 191	6·292 9028 — 10_n	3·342 487 — 10_n	1·987 501 — 10_n	9·037 081 — 20_n
3	0·000 007	5·684 498 — 10_n	2·328 363 — 10_n	1·899 306 — 10_n	8·543 171 — 20_n
4	0·000 120	4·948 864 — 10_n	1·102 374 — 10_n	0·905 494 — 10_n	7·059 004 — 20_n
s	0·005 018	0·121 718 — 10_n	1·045 843 — 20_n	0·422 333 — 10_n	1·346 458 — 20_n

In the numerical computation of the quantities occurring in the coefficients of the differential equations we regard the elements and other constants, that are made use of, as absolutely correct. It is then desirable to determine the number of decimal places of the several coefficients in order to make sure that the final result shall satisfy the degree of accuracy wanted.

Now in the theory of Jupiter's satellites the inclinations are known to 3 decimal places, if expressed in degrees, and it is not probable, that, in the near future, it will be possible appreciably to increase the accuracy of these quantities. Consequently it is quite sufficient to have the inclinations (expressed in our units) to 6 decimal places during a century; accordingly it is necessary to compute P_i and Q_i up to the sixth decimal place.

We have:

$$P_i = \Sigma_j \sigma_{ij} N_j \cos (b_j \tau + M_j)$$

$$Q_i = \Sigma_j \sigma_{ij} N_j \sin (b_j \tau + M_j)$$

We adopt, as a sufficient approximation, for the proper inclinations the values of de Sitter ("*Elements and Masses*" p. 594):

$$N_1 = 0°0272, \quad N_2 = 0°4683, \quad N_3 = 0°1839, \quad N_4 = 0°2536$$

We then find, expressed in our units:

$$N_1 = 0{\cdot}000\,4747\,, \quad N_2 = 0{\cdot}008\,173\,, \quad N_3 = 0{\cdot}003\,2097\,, \quad N_4 = 0{\cdot}004\,426$$

thus, the N_i relating to the satellites can be regarded as $< 0{\cdot}009$.

The inclination of Jupiter's orbit to its equator is 3°1153 (1900·0), whence we can assume that, expressed in our units, N_5 is about 0·054 372. In order to determine the number of decimal places of the quantities b_j and σ_{ij} we call the corrections, which must be applied to b_j and σ_{ij}, because of the uncertainty of numerical computations, ε_j and ε_{ij} respectively. Since the unit of time is 1^d122 and $\sigma_{ii} = 1$, the error of P_i and Q_i will then in a century increase to about $32500\, N_i \varepsilon_i$. Now in that interval P_i and Q_i must be accurate to 6 decimal places, so that $32500\, N_i \varepsilon_i < 10^{-6}$, whence $\varepsilon_i < 3 \times 10^{-9}$ and thus the b_i, — in the case of the four satellites —, must be computed to 9 decimal places.

In the case of b_5, it is also sufficient to determine it to 9 decimal places, because the correction ε_i must be smaller than $0{\cdot}6 \times 10^{-9}$.

By the same reasoning we find that the σ_{ij} must be accurate to 4 decimal places.

The b_i are the roots of equation (H) which is of the fifth degree. Supposing the latter to be written in the form:

$$b^5 + a\, b^4 + a_1\, b^3 + a_2\, b^2 + a_3\, b + a_4 = 0 \tag{48}$$

we have:

$$a = \Sigma_j\, b_j, \text{ where } j = 1, \ldots 4, 5, \text{ and } b_j \text{ is a root of (48)}$$

Hence:

$$\delta a = \Sigma_j\, \delta b_j$$

and, since $\delta b_j < 10^{-9}$, it is quite sufficient to compute a to 9 decimal places, even 8 decimals would suffice.

Now from equation (H) it follows, that:

$$a = \Sigma_i\, [i], \quad i = 1, \ldots 4, 5$$

Hence the coefficients $[i]$ must be computed to 9 decimal places.

Concerning the constants (ij) we can ascertain by a similar reasoning, to how many decimal places we need them.

By equation (48) we have:

$$-a_1 = \Sigma b_i b_j$$

and thus:

$$-\delta a_1 = \Sigma_j \delta b_j \, [\Sigma_i b_i] \tag{49}$$

where Σ_i relates to all values of i, except $i = j$.

It is very probable that the values for b_i given by Souillart (see *Souillart* II p. 156 and p. 157) are good approximations of our b_i. Adopting his values equation (49) assumes the form:

$$-\delta a_1 = 0{\cdot}0143 \, \delta b_j$$

hence:

$$-\delta a_1 < 0{\cdot}0143 \times 10^{-9}$$

thus a_1 must be computed to 11 decimal places.

We now have by equations (H) and (48):

$$a_1 = \Sigma (ij)(kl) \qquad (8 \text{ terms})$$

where, according to the tabular values on p. 59, the constants (ij) are about 0·00007 at the utmost.

If we denote the error of each of the last constants by $\delta_\varkappa$, we must have:

$$8 \times 0{\cdot}0001 \, \delta_\varkappa < 10^{-11}$$

and thus:

$$\delta_\varkappa < 10^{-8}$$

Hence the coefficients (ij) must be computed to 8 decimal places.

By a similar discussion of the other coefficients a_2, a_3, a_4 of (48) it appears that, if we compute $[i]$ to 9, and (ij) to 8 places, the conditions of all coefficients a_i are satisfied. In the computations of A_i, C_i, and B_{ij} we have taken account of these limits and then find:

i	A_i	$2\,C_i$	$\log C_i$
1	— 0·00079 27662	— 0·00000 00618	2·490 194 — 10_n
2	— 0·00019 65121	— 0·00000 00097	1·686 958 — 10_n
3	— 0·00004 83826	— 0·00000 00079	1·598 467 — 10_n
4	— 0·00000 88905	— 0·00000 00008	0·604 526 — 10_n
s	— 0·00000 000013	— 0·00000 00003	0·121 304 — 10_n

j	$\log 2\,B_{1j}$	j	$\log 2\,B_{2j}$	j	$\log 2\,B_{3j}$	j	$\log 2\,B_{4j}$
2	5·477 0706 — 10_n	1	5·674 5600 — 10_n	1	4·790 7846 — 10_n	1	3·959 2484 — 10_n
3	5·113 5339 — 10_n	3	5·789 5430 — 10_n	2	5·269 3043 — 10_n	2	4·234 1047 — 10_n
4	4·023 8024 — 10_n	4	4·496 1481 — 10_n	4	5·111 2087 — 10_n	3	5·369 4040 — 10_n
s	3·497 191 — 10_n	s	3·900 557 — 10_n	s	4·306 144 — 10_n	s	4·796 577 — 10_n

In the numerical computation of the quantities occurring in the coefficients of the differential equations we regard the elements and other constants, that are made use of, as absolutely correct. It is then desirable to determine the number of decimal places of the several coefficients in order to make sure that the final result shall satisfy the degree of accuracy wanted.

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we have:

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Hence:

$$\delta a = \Sigma_j\, \delta b_j$$

and, since $\delta b_j < 10^{-9}$, it is quite sufficient to compute a to 9 decimal places, even 8 decimals would suffice.

Now from equation (H) it follows, that:

$$a = \Sigma_i\, [i], \quad i = 1, \ldots 4, 5$$

Hence the coefficients $[i]$ must be computed to 9 decimal places.

Concerning the constants (ij) we can ascertain by a similar reasoning, to how many decimal places we need them.

By equation (48) we have:

$$-a_1 = \Sigma b_i b_j$$

and thus:

$$-\delta a_1 = \Sigma_j \delta b_j \, [\Sigma_i b_i] \tag{49}$$

where Σ_i relates to all values of i, except $i = j$.

It is very probable that the values for b_i given by Souillart (see *Souillart* II p. 156 and p. 157) are good approximations of our b_i. Adopting his values equation (49) assumes the form:

$$-\delta a_1 = 0{\cdot}0143 \, \delta b_j$$

hence:

$$-\delta a_1 < 0{\cdot}0143 \times 10^{-9}$$

thus a_1 must be computed to 11 decimal places.

We now have by equations (H) and (48):

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where, according to the tabular values on p. 59, the constants (ij) are about 0·00007 at the utmost.

If we denote the error of each of the last constants by δ_κ, we must have:

$$8 \times 0{\cdot}0001 \, \delta_\kappa < 10^{-11}$$

and thus:

$$\delta_\kappa < 10^{-8}$$

Hence the coefficients (ij) must be computed to 8 decimal places.

By a similar discussion of the other coefficients a_2, a_3, a_4 of (48) it appears that, if we compute $[i]$ to 9, and (ij) to 8 places, the conditions of all coefficients a_i are satisfied. In the computations of A_i, C_i, and B_{ij} we have taken account of these limits and then find:

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3	— 0·00004 83826	— 0·00000 00079	1·598 467 — 10_n
4	— 0·00000 88905	— 0·00000 00008	0·604 526 — 10_n
s	— 0·00000 000013	— 0·00000 00003	0·121 304 — 10_n

j	$\log 2\,B_{1j}$	j	$\log 2\,B_{2j}$	j	$\log 2\,B_{3j}$	j	$\log 2\,B_{4j}$
2	5·477 0706 — 10_n	1	5·674 5600 — 10_n	1	4·790 7846 — 10_n	1	3·959 2484 — 10_n
3	5·113 5339 — 10_n	3	5·789 5430 — 10_n	2	5·269 3043 — 10_n	2	4·234 1047 — 10_n
4	4·023 8024 — 10_n	4	4·496 1481 — 10_n	4	5·111 2087 — 10_n	3	5·369 4040 — 10_n
s	3·497 191 — 10_n	s	3·900 557 — 10_n	s	4·306 144 — 10_n	s	4·796 577 — 10_n

j	$2\,B_{1j}$	j	$2\,B_{2j}$	j	$2\,B_{3j}$	j	$2\,B_{4j}$
2	— 0·00002 99875	1	— 0·00604 72464	1	— 0·00000 61771	1	— 0·00000 09104
3	— 0·00001 29878	3	— 0·00006 15947	2	— 0·00001 85911	2	— 0·00000 17144
4	— 0·00000 10563	4	— 0·00000 31344	4	— 0·00001 29184	3	— 0·00002 34101
s	— 0·00000 03142	s	— 0·00000 07953	s	— 0·00000 20237	s	— 0·00000 62600

Tabular values of the constants $[i]$.

These constants are defined by the expression: $[i] = \frac{2}{G_i}\,[A_i + \Sigma_j B_{ij}] + \frac{2\,C_i}{Cn}$, where $j = 1, \ldots 4, s$, but $j \neq i$.

They thus involve the quantities G_i and n.

The constant G_i is the constant of areas: $G_i = a_i^2 n_i \sqrt{1 - e_i^2}$. The numerical values are:

$$\log G_1 = 9\cdot8000\,2856 - 10$$
$$\log G_2 = 9\cdot9007\,690 - 10$$
$$\log G_3 = 0\cdot0021\,635$$
$$\log G_4 = 0\cdot1247\,379$$

The constant n is the velocity of rotation of the planet round its axis; de Sitter gives for it *) as extreme values $n = 9^h\,50^m5$ and $n = 9^h\,55^m5$. Adopting for n the mean of the two we find, expressed in our units: $\log n = 1\cdot233\,533$.

In the following tables we have denoted the parts, of which $[i]$ consist by I and II.

i	I	II	$[i]$	$\log [i]$
1	— 0·00258 30123	— 0·00000 32743	— 0·00258 62873	$7\cdot4126\,7677 - 10_n$
2	— 0·00063 56316	— 0·00000 05151	— 0·00063 61467	$6\cdot8035\,573 - 10_n$
3	— 0·00013 57973	— 0·00000 04201	— 0·00013 62174	$6\cdot1342\,326 - 10_n$
4	— 0·00003 75743	— 0·00000 00426	— 0·00003 76169	$5\cdot5753\,830 - 10_n$

The coefficient $\{5\}$ of the fifth diagonal term in (H) is defined by the formula:

$$\{5\} = \frac{2\,C_s}{Cn}\cos i_0.$$

We find:

$$\{5\} = -\,0\cdot0000\,0001\,3982\,6 \qquad \log\{5\} = 2\cdot145\,589 - 10_n$$

The coefficients $[i]$ and $\{5\}$ must be good approximations of the roots b_i of equation (H). If we compare them with the values b_j of Souillart, there is a satisfactory agreement, except in the case b_5. It can therefore be expected that our b_i will not much differ from Souillart's.

Tabular values of the constants (ij).

We have the formula:

$$(ij) = -\frac{2\,B_{ij}}{\sqrt{G_i G_j}} + \frac{2\,C_j}{Cn}\sqrt{\frac{G_i}{G_j}}$$

where $j = 1, \ldots 4$.

*) Proceedings Royal Academy Amst. Vol. XXIII p. 1056.

In the case of the Sun we have:

$$(i\,5) = -\frac{2\,B_{is}}{\sqrt{G_i}} + \frac{2\,C_s}{Cn}\sqrt{G_i}$$

Denoting again the two parts of (ij) by the symbols I and II, we find:

j	I	II	$(1j)$	$\log(1j)$
2	+ 0·00004 23323	— 0·00000 04587	+ 0·00004 18736	5·621 9403 — 10
3	+ 0·00001 63094	— 0·00000 03329	+ 0·00001 59765	5·203 4817 — 10
4	+ 0·00000 11519	— 0·00000 00293	+ 0·00000 11226	4·050 2211 — 10
5	+ 0·00000 03955	— 0·00000 00111	+ 0·00000 03844	3·584 7879 — 10

j	I	II	$(2j)$	$\log(2j)$
1	+ 0·00006 67054	— 0·00000 36770	+ 0·00006 30284	5·799 5363 — 10
3	+ 0·00006 88774	— 0·00000 03739	+ 0·00006 85035	5·835 7128 — 10
4	+ 0·00000 30436	— 0·00000 00329	+ 0·00000 30107	4·478 6718 — 10
5	+ 0·00000 08916	— 0·00000 00125	+ 0·00000 08791	3·944 0645 — 10

j	I	II	$(3j)$	$\log(3j)$
1	+ 0·00000 77569	— 0·00000 41323	+ 0·00000 36246	4·559 2649 — 10
2	+ 0·00002 07892	— 0·00000 05789	+ 0·00002 02103	5·305 5728 — 10
4	+ 0·00001 11578	— 0·00000 00370	+ 0·00001 11208	5·046 1360 — 10
5	+ 0·00000 20186	— 0·00000 00140	+ 0·00000 20046	4·302 0299 — 10

j	I	II	$(4j)$	$\log(4j)$
1	+ 0·00000 09928	— 0·00000 47586	— 0·00000 37658	4·575 8527 — 10_n
2	+ 0·00000 17655	— 0·00000 06666	+ 0·00000 10989	4·040 9385 — 10
3	+ 0·00002 02280	— 0·00000 04838	+ 0·00001 97442	5·295 4395 — 10
5	+ 0·00000 54226	— 0·00000 00162	+ 0·00000 54064	4·732 9106 — 10

The constants $\{i\}$, defined by $\{i\} = \frac{2\,C_i \cos i_c}{Cn\sqrt{G_i}}$, are:

i	$\{i\}$	$\log\{i\}$
1	— 0·00000 41159	4·614 465 — 10_n
2	— 0·00000 05766	3·760 858 — 10_n
3	— 0·00000 04185	3·621 670 — 10_n
4	— 0·00000 00369	2·566 442 — 10_n

The equation (H).

Instead of b we introduce as unknown a quantity a defined by $a = -b \times 10^{-4}$. According to p. 56 it then appears that the coefficient of a^4 must be computed to 5 decimal places, while the other coefficients should be known to more places.*) With the preceding data we then find for the equation in a:

$$(H_1) \quad a^5 + 33{\cdot}962823\, a^4 + 220{\cdot}63023\, a^3 + 297{\cdot}67586\, a^2 + 79{\cdot}203225\, a + 0{\cdot}0281321 = 0$$

In order to determine the roots of this equation we can take as first approximations the values $[i] \times 10^{-4}$ and after a few successive approximations by Newton's method, we then find for the roots the following values, accurate to the last decimal place given:

$$-25{\cdot}876\,553, \quad -6{\cdot}375\,8514, \quad -1{\cdot}3566\,861, \quad -0{\cdot}3533\,76686, \quad -0{\cdot}0003\,556635.$$

Hence the values of the b_i are:

$$\begin{aligned} b_1 &= +\,0{\cdot}0025\,876553 \\ b_2 &= +\,0{\cdot}0006\,375851 \\ b_3 &= +\,0{\cdot}0001\,356686 \\ b_4 &= +\,0{\cdot}0000\,353377 \\ b_5 &= +\,0{\cdot}0000\,000355\,6635 \end{aligned}$$

As has been pointed out above the roots b_i — even disregarding the unavoidable uncertainties of logarithmic computation — are only trustworthy to the ninth decimal place. For the first four b_i this is entirely sufficient and an error in that place does not sensibly affect their values, but in the case of b_5 there would be an uncertainty of about $\frac{1}{36}$ of the absolute value. Therefore we were obliged to compute several more decimal places of b_5, and if we regard the value obtained as a pure numerical result, and suppose the coefficients of (H_1) to be exact, it is entirely trustworthy, since the error has been determined.

For the computations we have for the greater part used 6 figure tables, and when necessary 7 figures, in a few cases the 8 figure tables of Peters.

As will soon appear these values of b_i do not differ much from those of Souillart-de Sitter, in the case of $i = 1, 2, 3$ the difference does not exceed one hundredth part of their amount, in the case of b_4 the difference is about $\frac{1}{50}$ of the value, while a comparison of b_5 was not possible for reasons which will be explained below.

The equations for σ_{ij}.

We have now to determine the constants σ_{ij} from the sets of equations:

$$(G) \qquad \begin{array}{l} \big[b_j + [1]\big]\sigma_{1j} + (12)\,\sigma_{2j} + (13)\,\sigma_{3j} + (14)\,\sigma_{4j} + (15)\,\sigma_{5j} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (41) \quad \sigma_{1j} + (42)\,\sigma_{2j} + (43)\,\sigma_{3j} + \big[b_j + [4]\big]\sigma_{4j} + (45)\,\sigma_{5j} = 0 \end{array}$$

where $\sigma_{jj} = 1$.

The σ_{ij} have been computed from these equations by successive approximations. The values found have been checked by substituting them in the equations and the approximations have been carried on until the equations were exactly satisfied by the σ_{ij}. Moreover, as a strong check not only on the σ_{ij} but also on the b_j, the values of σ_{ij} have been substituted in the fifth equation (G), which had not been used for the computation of the σ_{ij}. It was found that these equations also were very well satisfied. The values thus found are:

*) In consequence of the accuracy of the tables of logarithms used, it was not possible to compute the coefficients to more places than are given above.

$b_1 = + 0{\cdot}0025\,876553$

$\sigma_{11} = + 1$
$\sigma_{21} = - 0{\cdot}032\,2257$
$\sigma_{31} = - 0{\cdot}001\,2208$
$\sigma_{41} = + 0{\cdot}001\,4974$
$\sigma_{51} = + 0{\cdot}001\,3825$

$b_2 = + 0{\cdot}0006\,375851$

$\sigma_{12} = + 0{\cdot}0211\,568$
$\sigma_{22} = + 1$
$\sigma_{32} = - 0{\cdot}0404\,579$
$\sigma_{42} = - 0{\cdot}0003\,76868$
$\sigma_{52} = + 0{\cdot}0010\,2268$

$b_3 = + 0{\cdot}0001\,356686$

$\sigma_{13} = + 0{\cdot}008\,7611$
$\sigma_{23} = + 0{\cdot}036\,767$
$\sigma_{33} = + 1$
$\sigma_{43} = - 0{\cdot}202\,779$
$\sigma_{53} = + 0{\cdot}004\,012$

$b_4 = + 0{\cdot}0000\,353377$

$\sigma_{14} = + 0{\cdot}0014\,526$
$\sigma_{24} = + 0{\cdot}0181\,630$
$\sigma_{34} = + 0{\cdot}1139\,74$
$\sigma_{44} = + 1$
$\sigma_{54} = + 0{\cdot}0026\,6102$

$b_5 = + 0{\cdot}0000\,0003\,5566\,35$

$\sigma_{15} = + 0{\cdot}0004\,7865$
$\sigma_{25} = + 0{\cdot}0052\,5250$
$\sigma_{35} = + 0{\cdot}0284\,907$
$\sigma_{45} = + 0{\cdot}1589\,339$
$\sigma_{55} = + 1$

In order to compare the preceding results with those of Souillart's theory we start from the formulae derived by de Sitter in *Gron. Publ.* 17 p. 86, and also given in "*Elements and masses*" pp. 722 and 723, expressing the b_j and σ_{ij} in terms of the masses and the constant of compression of the planet. According to de Sitter's statements the corrections db_j and $d\sigma_{ij}$ there derived correspond to Souillart's first method of integration, which is not a legitimate one. The final expressions of b_j and σ_{ij} were then found by adding the corrections db_j and $d\sigma_{ij}$ to the b_j and σ_{ij} of the second method of integration. The main part of the b_j and σ_{ij} of course consists of the original b_j and σ_{ij} of the latter method, which is the same as that employed by us. Although in consequence of this process it cannot be expected that the values thus derived from Souillart's theory should completely agree with the above, we undoubtedly can assume them to be better than the uncorrected values of Souillart himself, which involve another set of constants. There is still another reason why the agreement cannot be complete viz. that the plane of (x, y) in both methods is different and the equations thus have another meaning. (See chapter V).

The expressions of b_j and σ_{ij} given by de Sitter in 1908 are:

$b_1 = - 0^{\circ}13614 - 0{\cdot}1327\ \rho - 0{\cdot}0023\ \lambda_2 - 0{\cdot}0010\ \lambda_3 - 0{\cdot}00008\ \lambda_4$
$b_2 = - 0{\cdot}032\,335 - {\cdot}02602\ \rho - {\cdot}00198\ \lambda_1 - {\cdot}00013\ \lambda_2 - {\cdot}00399\ \lambda_3 - 0{\cdot}000191\ \lambda_4$
$b_3 = - 0{\cdot}006\,854 - {\cdot}00493\ \rho - {\cdot}00021\ \lambda_1 - {\cdot}00071\ \lambda_2 - {\cdot}00004\ \lambda_3 - {\cdot}000695\ \lambda_4$
$b_4 = - 0{\cdot}001\,772 - {\cdot}00077\ \rho - {\cdot}00003\ \lambda_1 - {\cdot}00007\ \lambda_2 - {\cdot}00075\ \lambda_3 + {\cdot}000098\ \lambda_4$

$\sigma_{21} = - 0{\cdot}0190 + 0{\cdot}012\ \rho - 0{\cdot}019\ \lambda_1$
$\sigma_{31} = - 0{\cdot}001 + {\cdot}001\ \rho - {\cdot}001\ \lambda_1$
$\sigma_{41} = 0{\cdot}000$

$\sigma_{12} = + 0{\cdot}0203 - 0{\cdot}020\ \rho + 0{\cdot}020\ \lambda_2$
$\sigma_{32} = - 0{\cdot}0347 + {\cdot}028\ \rho + {\cdot}002\ \lambda_1 - 0{\cdot}035\ \lambda_2 + 0{\cdot}005\ \lambda_3 - 0{\cdot}0005\ \lambda_4$
$\sigma_{42} = - 0{\cdot}0010 - {\cdot}001\ \rho - {\cdot}001\ \lambda_2 + {\cdot}001\ \lambda_3$

$$\sigma_{13} = +\,0{\cdot}0056 \;-\; 0{\cdot}013\,\rho \;+\; 0{\cdot}003\,\lambda_2 \;+\; 0{\cdot}010\,\lambda_2 \;-\; 0{\cdot}0001\,\lambda_4$$
$$\sigma_{23} = +\,0{\cdot}1488 \;+\; {\cdot}132\,\rho \;-\; {\cdot}011\,\lambda_1 \;+\; {\cdot}005\,\lambda_2 \;+\; {\cdot}125\,\lambda_3 \;+\; 0{\cdot}0026\,\lambda_4$$
$$\sigma_{43} = -\,0{\cdot}1772 \;+\; {\cdot}176\,\rho \;+\; {\cdot}008\,\lambda_1 \;+\; {\cdot}028\,\lambda_2 \;-\; {\cdot}211\,\lambda_3 \;+\; {\cdot}0282\,\lambda_4$$

$$\sigma_{14} = -\,0{\cdot}0018 \;-\; 0{\cdot}003\,\rho \;+\; 0{\cdot}001\,\lambda_3 \;+\; 0{\cdot}0018\,\lambda_4$$
$$\sigma_{24} = +\,{\cdot}0183 \;-\; {\cdot}034\,\rho \;-\; {\cdot}002\,\lambda_1 \;-\; {\cdot}002\,\lambda_2 \;+\; 0{\cdot}017\,\lambda_3 \;+\; 0{\cdot}0207\,\lambda_4$$
$$\sigma_{34} = +\,{\cdot}1203 \;-\; {\cdot}110\,\rho \;-\; {\cdot}005\,\lambda_1 \;-\; {\cdot}016\,\lambda_2 \;+\; {\cdot}021\,\lambda_3 \;+\; {\cdot}1064\,\lambda_4$$

The quantities ρ and λ_i are defined by: $Jb^2 = (Jb^2)_0\,(1+\rho)$, $m_i = (m_i)_0\,(1+\lambda_i)$ where:

$$(Jb^2)_0 = 0{\cdot}0000\,0000\,518$$
$$(m_1)_0 = 0{\cdot}0000\,260$$
$$(m_2)_0 = 0{\cdot}0000\,231$$
$$(m_3)_0 = 0{\cdot}0000\,804$$
$$(m_4)_0 = 0{\cdot}0000\,425$$

whence, for the constants adopted by us, whe have:

$$\rho = -\,0{\cdot}03506 \;,\quad \lambda_1 = 0{\cdot}500 \;,\quad \lambda_2 = 0{\cdot}0714 \;,\quad \lambda_3 = 0{\cdot}0199 \;,\quad \lambda_4 = 0{\cdot}0653.$$

Then we find:

$$b_1 = -\,0{\cdot}0025\,79$$
$$b_2 = -\,0{\cdot}0006\,385$$
$$b_3 = -\,0{\cdot}0001\,348$$
$$b_4 = -\,0{\cdot}0000\,3475$$

(It should be remembered that Souillart's b_j and ours have opposite signs)

$$\sigma_{21} = -\,0{\cdot}029 \qquad \sigma_{12} = +\,0{\cdot}0224 \qquad \sigma_{13} = +\,0{\cdot}0065 \qquad \sigma_{14} = -\,0{\cdot}0016$$
$$\sigma_{31} = -\,0{\cdot}001^5 \qquad \sigma_{32} = -\,0{\cdot}0371 \qquad \sigma_{23} = +\,0{\cdot}1425 \qquad \sigma_{24} = +\,0{\cdot}0201$$
$$\sigma_{41} = +\,0{\cdot}000 \qquad \sigma_{42} = -\,0{\cdot}0010 \qquad \sigma_{43} = -\,0{\cdot}1797 \qquad \sigma_{34} = +\,0{\cdot}1289$$

These tables show that the agreement, as concerns the b_i, is satisfactory, but in the case of σ_{ij} it is not all that could be desired. However — as has already been mentioned — we cannot directly compare the results, and it is clear that, if they disagree, the discrepancies will be more sensible in the inclinations than in the nodes, i. e. more in the values of the σ_{ij} then in those of the b_j.

Numerical integration of the equation for σ_j.

Since the argument $[A_i]$ in the integral:

$$\sigma_i = [A_i]\,\tau + C_i$$

contains the proper inclinations N_i and thus depends on a special set of initial values it is convenient to tabulate separately the two factors of $[A_i]$, one containing the N_i and the other not containing it. We therefore put:

$$[A_i] = k_i\,N_i^2$$

where:

$$k_i = -\frac{1}{G_i^2}\,(A_i + \Sigma_j B_{ij})$$

and we have:

i	log k_i	log N_i^2
1	7·311 06 — 10	2·961 59 — 10
2	6·601 40 — 10	5,743 38 — 10
3	5·829 69 — 10	5·032 65 — 10
4	5·149 12 — 10	5·426 20 — 10

We then find:

$$\sigma_1 = 0{\cdot}00000\ 00001\ 87\ \tau.$$
$$\sigma_2 = 0{\cdot}00000\ 00221\ 19\ \tau.$$
$$\sigma_3 = 0{\cdot}00000\ 00067\ 28\ \tau.$$
$$\sigma_4 = 0{\cdot}00000\ 00003\ 76\ \tau.$$

The integration-constants C_i must be taken equal to zero, since for $\tau = 0$ we must have $\sigma_i = 0$.

CHAPTER IV.

DEVELOPMENTS OF THE b_j AND σ_{ij} IN TERMS OF THE MASSES.

The values of the roots b_j and of the coefficients σ_{ij} depend on the masses, the constants of compression J and K and on other elements, especially on the mean distances, of which the last can be regarded as accurately known. Although the values adopted here for the masses and the constants J and K are certainly nearer the truth than those of Souillart, still de Sitter's opinion given in *Gron. Publ.* 17 p. 50 applies also to our present results, viz: "The assumption necessarily underlying any practical determination of corrections to the adopted values of the masses and elements on which a theory is constructed is that only the more important terms are liable to correction and that these corrections are simple *linear* functions of the corrections to the elements and masses. In the case of Jupiter's satellites, however, the uncertainty of the adopted masses is so large, compared with the accuracy of modern observations, that in most cases this assumption is no longer admissible". Prof. de Sitter then gives the outlines of a method by which it is possible partially to overcome these difficulties and to find developments suitable for numerical evaluation.

In the following we have endeavoured to apply this method and derive developments of b_i and σ_{ij}.

In order to introduce the corrections of the masses and other elements, we start from the general equations:

$$\left.\begin{aligned} m_i &= m_{io}\,(1 + \nu_i) \\ J_i &= J_{io}\,(1 + \mu_o) \\ K_i &= K_{io}\,(1 + \mu_5) \\ Cu &= Cu_o\,(1 + \eta) \end{aligned}\right\} \qquad i = 1, \ldots, 4$$

where m_{io}, J_{io}, K_{io}, Cu_o are the values adopted in the preceding, while ν_i, μ_o, μ_5 and η are the corresponding corrections. Thus e. g.:

$$\nu_i = \frac{\Delta m_{io}}{m_{io}}$$

If we compare our values of m_i, J_i with the values of these elements in "*Elements and Masses*", p. 718 we can assume, taking account of the probable errors there given, :.

$$\nu_1 < \tfrac{1}{2}, \qquad \nu_2 < \tfrac{1}{10}, \qquad \nu_3 < \tfrac{1}{10}, \qquad \nu_4 < \tfrac{1}{4}, \qquad \mu_o < \tfrac{1}{20}.$$

As to the other corrections prof. de Sitter concluded from his investigations that the corrections to be applied to the adopted values of K and C wil not exceed about $\frac{1}{3}$ and $\frac{1}{4}$ resp. of the true value; therefore we can put $\mu_5 < \frac{1}{3}$ and $\eta < \frac{1}{4}$.

The coefficients of the equations from which b_j and σ_{ij} are computed depend in a very intricate way on the masses and elements. Each of these coefficients must be developed in terms of the corrections ν_i etc. In these developments and also in that of the corrections Δb_j and $\Delta\sigma_{ij}$ all products and powers of the corrections ν_i etc. were neglected, if they were smaller than 0·01. This limit is somewhat arbitrary, as appears by an inspection of the developments, since on the one hand very small corrections were taken into account, while on the other hand rather large ones were omitted. The last difficulty has been compensated where necessary by taking them into account in the most important cases.

It is convenient to recall the expressions of the different coefficients in equations (G). They are:

$$[i] = \frac{2}{G_i}\,[A_i + \Sigma_j\, B_{ij}] + \frac{2\,C_i}{Cn}, \quad i = 1, \ldots . 4.$$

$$(ij) = -\frac{2\,B_{ij}}{\sqrt{G_i\,G_j}} + \frac{2\,C_j}{Cn}\sqrt{\frac{G_i}{G_j}}$$

$$\{i\} = \frac{2\,C_i\,\cos i_o}{Cn\,\sqrt{G_i}}$$

$$\{5\} = \frac{2\,C_s}{Cn}\,\cos i_o$$

whence we have:

$$\Delta\,[i] = \frac{2}{G_i}\,[\Delta\,A_i + \Sigma\,\Delta\,B_{ij}] + 2\,\Delta\,\frac{C_i}{Cn}$$

$$\Delta\,(ij) = -\frac{2\,\Delta\,B_{ij}}{\sqrt{G_i\,G_j}} + 2\sqrt{\frac{G_i}{G_j}}\,\Delta\,\frac{C_i}{Cn}$$

$$\Delta\,\{i\} = +\frac{2\,\cos i_o}{\sqrt{G_i}}\,\Delta\,\frac{C_i}{Cn}$$

$$\Delta\,\{5\} = 2\,\cos i_o\,\Delta\,\frac{C_s}{Cn}$$

Further we make use of the following developments:

$$A_i^{\mathrm{I}} = -\frac{1}{2}\,\frac{J_{io}}{a_i}\left(1 + \frac{3}{2}\,e_i^2\right)\,[1 + m_i\,(1 + \nu_i)]\,(1 + \mu_o) = A_{io}^{\mathrm{I}}\,(1 + \mu_o)$$

$$A_i^{\mathrm{II}} = \qquad\qquad = A_{io}^{\mathrm{II}}\,(1 + \mu_5)$$

$$B_{ij} = \qquad\qquad = B_{ijo}\,(1 + \nu_j)$$

$$C_i^{\mathrm{I}} = C_{io}^{\mathrm{I}}\,(1 + \nu_i + \mu_o + \nu_i\,\mu_o), \text{ where } \nu_i\,\mu_o = 0, \text{ if } i = 2, 3.$$

$$C_i^{\mathrm{II}} = C_{io}^{\mathrm{II}}\,(1 + \nu_i + \mu_5 + \nu_i\,\mu_5)$$

$$\frac{C_i^{\mathrm{I}}}{Cn} = \frac{C_{io}^{\mathrm{I}}}{Cn_o}\,[1 + \nu_i + \mu_o + \nu_i\,\mu_o - \eta - \nu_i\,\eta - \mu_o\,\eta + \eta^2 + \nu_i\,\eta^2 - \eta^3], \text{ where } \nu_i\,\mu_o = 0, \nu_i\,\eta^2 = 0, \text{ if } i = 2, 3.$$

$$\frac{C_i^{\mathrm{II}}}{Cn} = \frac{C_{io}^{\mathrm{II}}}{Cn_o}\,[1 + \nu_i + \mu_5 + \nu_i\mu_5 - \eta - \nu_i\eta - \mu_5\eta - \nu_i\mu_5\eta + \eta^2 + \nu_i\eta^2 + \mu_5\eta^2 + \nu_i\mu_5\eta^2 - \eta^3],$$

where $\nu_i\eta^2 = 0$, $\nu_i\mu_5\eta = 0$, $\nu_i\mu_5\eta^2 = 0$, if $i = 2, 3$.

Because of the small value of $\frac{C_{i_0}^{II}}{Cn_0}$ the last development is only of importance for $i = 1$, in all the other cases it can be neglected.

By means of these expressions we then find:

$$\Delta[1] = -[0{\cdot}0025\ 0889\ \mu_0 + 0{\cdot}0000\ 07117\ \mu_5 + 0{\cdot}0000\ 03274\ \nu_1 + 0{\cdot}0000\ 47538\ \nu_2 + 0{\cdot}0000\ 2058\ \nu_3 + 0{\cdot}0000\ 01674\ \nu_4 + 0{\cdot}0000\ 03265\ (\nu_1\mu_0 - \mu_0\eta + \eta^2 + \nu_1\eta^2 - \eta^3) + 0{\cdot}0000\ 03274\ (-\eta - \nu_1\eta) + 0{\cdot}0000\ 00009\ (\nu_1\mu_5 - \mu_5\eta)]$$

$$\Delta[2] = -[0{\cdot}0004\ 9388\ \mu_0 + 0{\cdot}0000\ 00553\ \mu_5 + 0{\cdot}0000\ 59401\ \nu_1 + 0{\cdot}0000\ 0052\ \nu_2 + 0{\cdot}0000\ 77406\ \nu_3 + 0{\cdot}0000\ 03939\ \nu_4 + 0{\cdot}0000\ 00515\ (-\eta - \nu_2\eta - \mu_0\eta + \eta^2 - \eta^3)]$$

$$\Delta[3] = -0{\cdot}0000\ 96662\ \mu_0 + 0{\cdot}0000\ 00042\ \mu_5 + 0{\cdot}0000\ 06146\ \nu_1 + 0{\cdot}0000\ 18499\ \nu_2 + 0{\cdot}0000\ 00420\ \nu_3 + 0{\cdot}0000\ 12854\ \nu_4 + 0{\cdot}0000\ 00420\ (-\eta - \nu_3\eta - \mu_0\eta + \eta^2 - \eta^3)]$$

$$\Delta[4] = -[0{\cdot}0000\ 13383\ \mu_0 + 0{\cdot}0000\ 00002\ \mu_5 + 0{\cdot}0000\ 00683\ \nu_1 + 0{\cdot}0000\ 012856\ \nu_2 + 0{\cdot}00001\ 7566\ \nu_3 + 0{\cdot}00000\ 0043\ (\nu_4 + \nu_4\mu_0 - \eta - \nu_4\eta - \mu_0\eta + \eta^2 + \nu_4\eta^2 - \eta^3)]$$

$$\Delta\{5\} = -0{\cdot}0000\ 00014\ [\mu_0 - \eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta(12) = +0{\cdot}0000\ 4187\ \nu_2 - 0{\cdot}0000\ 0046\ [\mu_0 - \eta - \nu_2\eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta(13) = +0{\cdot}0000\ 1598\ \nu_3 - 0{\cdot}0000\ 0033\ [\mu_0 - \eta - \nu_3\eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta(14) = +0{\cdot}0000\ 0112\ \nu_4 - 0{\cdot}0000\ 00029\ [\mu_0 + \nu_4\mu_0 - \eta - \nu_4\eta - \mu_0\eta + \eta^2 + \nu_4\eta^2 - \eta^3]$$

$$\Delta(15) = -0{\cdot}0000\ 00011\ [\mu_0 - \eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta\{1\} = -0{\cdot}0000\ 04116\ [\nu_1 + \mu_0 + \nu_1\mu_0 - \eta - \nu_1\eta - \mu_0\eta + \eta^2 + \nu_1\eta^2 - \eta^3]$$

$$\Delta(21) = +0{\cdot}0000\ 63028\ \nu_1 - 0{\cdot}0000\ 03677\ [\mu_0 + \nu_1\mu_0 - \eta - \nu_1\eta - \mu_0\eta + \eta^2 + \nu_1\eta^2 - \eta^3]$$

$$\Delta(23) = +0{\cdot}0000\ 68503\ \nu_3 - 0{\cdot}0000\ 00374\ [\mu_0 - \eta - \nu_3\eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta(24) = +0{\cdot}0000\ 03011\ \nu_4 - 0{\cdot}0000\ 00033\ [\mu_0 + \nu_4\mu_0 - \eta - \nu_4\eta - \mu_0\eta + \eta^2 + \nu_4\eta^2 - \eta^3]$$

$$\Delta(25) = -0{\cdot}0000\ 00012\ [\mu_0 - \eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta\{2\} = -0{\cdot}0000\ 00577\ [\nu_2 + \mu_0 - \eta - \nu_2\eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta(31) = +0{\cdot}0000\ 03625\ \nu_1 - 0{\cdot}0000\ 04132\ [\mu_0 + \nu_1\mu_0 - \eta - \nu_1\eta - \mu_0\eta + \eta^2 + \nu_1\eta^2 - \eta^3]$$

$$\Delta(32) = +0{\cdot}0000\ 20210\ \nu_2 - 0{\cdot}0000\ 00579\ [\mu_0 - \eta - \nu_2\eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta(34) = +0{\cdot}0000\ 11121\ \nu_4 - 0{\cdot}0000\ 00037\ [\mu_0 + \nu_4\mu_0 - \eta - \nu_4\eta - \mu_0\eta + \eta^2 + \nu_4\eta^2 - \eta^3]$$

$$\Delta(35) = -0{\cdot}0000\ 00014\ [\mu_0 - \eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta\{3\} = -0{\cdot}0000\ 00418\ [\nu_3 + \mu_0 - \eta - \nu_3\eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta(41) = -0{\cdot}0000\ 03766\ \nu_1 - 0{\cdot}0000\ 04759\ [\mu_0 + \nu_1\mu_0 - \eta - \nu_1\eta - \mu_0\eta + \eta^2 + \nu_1\eta^2 - \eta^3]$$

$$\Delta(42) = +0{\cdot}0000\ 01098\ \nu_2 - 0{\cdot}0000\ 00667\ [\mu_0 - \eta - \nu_2\eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta(43) = +0{\cdot}0000\ 19744\ \nu_3 - 0{\cdot}0000\ 00484\ [\mu_0 - \eta - \nu_3\eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta(45) = -0{\cdot}0000\ 00016\ [\mu_0 - \eta - \mu_0\eta + \eta^2 - \eta^3]$$

$$\Delta\{4\} = -0{\cdot}0000\ 00037\ [\nu_4 + \mu_0 + \nu_4\mu_0 - \eta - \nu_4\eta - \mu_0\eta + \eta^2 + \nu_4\eta^2 - \eta^3].$$

The motions of the nodes b_i and σ_{ij} have been determined from:

$$(50)\qquad \begin{aligned} &\big[b_1 + [1]\big]\,\sigma_{1i} + (12)\,\sigma_{2i} + (13)\,\sigma_{3i} + (14)\,\sigma_{4i} + (15)\,\sigma_{5i} = 0 \\ &\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &(41)\,\sigma_{1i} + (42)\,\sigma_{2i} + (43)\,\sigma_{3i} + \big[b_4 + [4]\big]\,\sigma_{4i} + (45)\,\sigma_{5i} = 0 \end{aligned}$$

If we introduce in these equations instead of $[i]$ and (ij) the corrected values $[i] + \Delta[i]$ and $(ij) + \Delta(ij)$, we find a system of equations for the determination of the corrected values $b_i + \Delta b_i$ and $\sigma_{ij} + \Delta\sigma_{ij}$, viz:

$$\left[b_i + \Delta b_i + [i] + \Delta[i]\right]\left[\sigma_{1i} + \Delta\sigma_{1i}\right] + \left[(12) + \Delta(12)\right]\left[\sigma_{2i} + \Delta\sigma_{2i}\right] + \left[(13) + \Delta(13)\right]\left[\sigma_{3i} + \Delta\sigma_{3i}\right]$$
$$+ \left[(14) + \Delta(14)\right]\left[\sigma_{4i} + \Delta\sigma_{4i}\right] + \left[(15) + \Delta(15)\right]\left[\sigma_{5i} + \Delta\sigma_{5i}\right] = 0$$

(51) etc.

$$\left[\}1\{ + \Delta\}1\{\right]\left[\sigma_{1i} + \Delta\sigma_{1i}\right] + \left[\}2\{ + \Delta\}2\{\right]\left[\sigma_{2i} + \Delta\sigma_{2i}\right] + \left[\}3\{ + \Delta\}3\{\right]\left[\sigma_{3i} + \Delta\sigma_{3i}\right]$$
$$+ \left[\}4\{ + \Delta\}4\{\right]\left[\sigma_{4i} + \Delta\sigma_{4i}\right] + \left[b_i + \Delta b_i + \}5\{ + \Delta\}5\{\right]\left[\sigma_{5i} + \Delta\sigma_{5i}\right] = 0.$$

It can at once be seen that it is almost impracticable to derive from the equations (51) the corrected values $b_i + \Delta b_i$ and $\sigma_{ij} + \Delta\sigma_{ij}$ and some approximative method must be used. Now if we regard the corrections Δb_i, $\Delta\sigma_{ij}$ as quantities of the first order of smallness and neglect quantities of the second order, it is possible to derive from (51) by means of (50) equations that are linear in Δb_i and $\Delta\sigma_{ij}$ and enable us to derive the developments of Δb_i and $\Delta\sigma_{ij}$. This method of procedure has not been followed here, since it was not certain that the hypothesis made that Δb_i and $\Delta\sigma_{ij}$ would be very small quantities would be confirmed by the computations, and if they were not, it would have been necessary to take into account Δb_i^2, etc., which would make the computation very intricate.

It was found better to introduce other unknown quantities x_i and y_{ij} instead of Δb_i and $\Delta\sigma_{ij}$, defined by:

$$x_i = \Delta b_i - \Delta[i] \quad , \quad y_{ji} = \Delta\sigma_{ji} - \frac{\Delta(ij)}{\sigma_{ji}}$$

if $i = 1, \ldots 4$ and:

$$x_5 = \Delta b_5 - \Delta\}5\{ \quad , \quad y_{j5} = \Delta\sigma_{j5} - \frac{\Delta\}i\{}{\sigma_{j5}}$$

On resolving then the systems of linear equations in x_i and y_{ji}, obtained by neglecting x_i^2, $x_i y_{ji}$, y_{ji}^2, etc. We found very small values for x_i and y_{ji}.

Moreover it is easily verified by means of the following equations, that the equations for x_i and y_{ji} have exactly the same coefficients as the equations for Δb_i and $\Delta\sigma_{ji}$; only the right hand sides differ, whence it is easy to find, by means of the results of one of the two systems, the roots of the other one, but it is obvious that the values found for Δb_i and $\Delta\sigma_{ji}$ from the latter set of equations will not be so accurate, since on the right hand sides quantities such as $\Delta^2 b_i$, $\Delta b_i \Delta\sigma_{ji}$ etc. have been neglected, and these are much larger than the quantities x_i^2 etc., which have been omitted in the former method, which is used below. After the computation had been made it could be verified that the effect of the rejection of these terms did not exceed the uncertainties of the computations, so that it could be neglected.

The equations have the form, in the case $i = 1$:

$$\begin{array}{lllllll}
x_1 + & (12)\, y_{21} + & (13)\, y_{31} + & (14)\, y_{41} + & (15)\, y_{51} = p_{11} \\
\sigma_{21}\, x_1 + & \left[b_1 + [2]\right] y_{21} + & (23)\, y_{31} + & (24)\, y_{41} + & (25)\, y_{51} = p_{21} \\
\sigma_{31}\, x_1 + & (32)\, y_{21} + & \left[b_1 + [3]\right] y_{31} + & (34)\, y_{41} + & (35)\, y_{51} = p_{31} \\
\sigma_{41}\, x_1 + & (42)\, y_{21} + & (43)\, y_{31} + & \left[b_1 + [4]\right] y_{41} + & (45)\, y_{51} = p_{41} \\
\sigma_{51}\, x_1 + & \}2\{\, y_{21} + & \}3\{\, y_{31} + & \}4\{\, y_{41} + & \left[b_1 + \}5\{\right] y_{51} = p_{51}
\end{array} \qquad (52)$$

where:

$$p_{11} = -2\Delta[1] - \left[\sigma_{21} + \frac{(12)}{\sigma_{21}}\right]\Delta(12) - \left[\sigma_{31} + \frac{(13)}{\sigma_{31}}\right]\Delta(13) - \left[\sigma_{41} + \frac{(14)}{\sigma_{41}}\right]\Delta(14) - \left[\sigma_{51} + \frac{(15)}{\sigma_{51}}\right]\Delta(15) - \frac{\Delta^2(12)}{\sigma_{21}} - \frac{\Delta^2(13)}{\sigma_{31}} - \frac{\Delta^2(14)}{\sigma_{41}} - \frac{\Delta^2(15)}{\sigma_{51}}$$

$$p_{21} = -\Delta(21) - \sigma_{21}\left[\Delta[1]+\Delta[2]\right] - \left[b_1+[2]\right]\frac{\Delta(12)}{\sigma_{21}} - \left[\Delta[1]+\Delta[2]\right]\frac{\Delta(12)}{\sigma_{21}} - (23)\frac{\Delta(13)}{\sigma_{31}} - \sigma_{31}\Delta(23) - \frac{\Delta(23)\Delta(13)}{\sigma_{31}} - (24)\frac{\Delta(14)}{\sigma_{41}} - \sigma_{41}\Delta(24) - \frac{\Delta(24)\Delta(14)}{\sigma_{41}} - (25)\frac{\Delta(15)}{\sigma_{51}} - \sigma_{51}\Delta(25) - \frac{\Delta(25)\Delta(15)}{\sigma_{51}}$$

and similar equations for p_{31} and p_{41}, while:

$$p_{51} = -\Delta\{1\} - \{2\}\frac{\Delta(12)}{\sigma_{21}} - \sigma_{21}\Delta\{2\} - \frac{\Delta\{2\}\Delta(12)}{\sigma_{21}} - \{3\}\frac{\Delta(13)}{\sigma_{31}} - \sigma_{31}\Delta\{3\} - \frac{\Delta\{3\}\Delta(13)}{\sigma_{31}} - \{4\}\frac{\Delta(14)}{\sigma_{41}} - \sigma_{41}\Delta\{4\} - \frac{\Delta\{4\}\Delta(14)}{\sigma_{41}} - b_1\frac{\Delta(15)}{\sigma_{51}} - \Delta[1]\sigma_{51} - \frac{\Delta[1]\Delta(15)}{\sigma_{51}} - \frac{\{5\}\Delta(15)}{\sigma_{51}} - \sigma_{51}\Delta\{5\} - \frac{\Delta\{5\}\Delta(15)}{\sigma_{51}}$$

and similar expressions for the other systems p_{ji}

The equations (52) were then first resolved, retaining p_{ji} in the expressions of x_i and y_{ji}. As a check on the computations the conditions were employed that 1°: substituting the expressions for x_i and y_{ji} in the equations (52) we must find, say in the first equation for the coefficient of p_{11} unity, while the coefficients of the other p_{j1} must vanish. It appeared that in all the cases this condition was almost exactly satisfied.

2° Adding each set of five equations we must find for the left-hand sides: $\sum p_{ji}$. In almost all the cases this condition was nearly exactly fulfilled.

We thus found the following values of x_i and y_{ji}:

$$\begin{aligned}
y_{21} &= \ \ 0{\cdot}001\,65\ p_{11}' + 0{\cdot}051\,22\ p_{21}' - 0{\cdot}001\,45\ p_{31}' - 0{\cdot}000\,03\ p_{41}' - 0{\cdot}000\,002\ p_{51}'\\
y_{31} &= \ \ 0{\cdot}000\,03\ p_{11}' - 0{\cdot}000\,42\ p_{21}' + 0{\cdot}041\,03\ p_{31}' - 0{\cdot}000\,86\ p_{41}' - 0{\cdot}000\,41\ p_{51}'\\
y_{41} &= -0{\cdot}000\,058\ p_{11}' - 0{\cdot}000\,018\ p_{21}' - 0{\cdot}000\,31\ p_{31}' + 0{\cdot}039\,21\ p_{41}' - 0{\cdot}000\,58\ p_{51}'\\
y_{51} &= -0{\cdot}000\,054\ p_{11}' + 0{\cdot}000\,012\ p_{21}' + 0{\cdot}000\,006\ p_{31}' + 0{\cdot}000\,02\ p_{41}' + 0{\cdot}038\,66\ p_{51}'\\
x_1 &= +0{\cdot}000\,0999\ p_{11}' - 0{\cdot}000\,0021\ p_{21}' + 0{\cdot}000\,0006\ p_{31}' - 0{\cdot}000\,0000\ p_{41}' - 0{\cdot}000\,000\ p_{51}'
\end{aligned}$$

$$\begin{aligned}
y_{12} &= -0{\cdot}051\,35\ p_{12}' + 0{\cdot}001\,150\ p_{22}' + 0{\cdot}001\,55\ p_{32}' + 0{\cdot}000\,107\ p_{42}' + 0{\cdot}0000\,226\ p_{52}'\\
y_{32} &= +0{\cdot}000\,63\ p_{12}' + 0{\cdot}008\,02\ p_{22}' + 0{\cdot}198\,49\ p_{32}' - 0{\cdot}003\,64\ p_{42}' - 0{\cdot}000\,60\ p_{52}'\\
y_{42} &= -0{\cdot}000\,34\ p_{12}' - 0{\cdot}000\,19\ p_{22}' - 0{\cdot}006\,53\ p_{32}' + 0{\cdot}166\,80\ p_{42}' - 0{\cdot}001\,39\ p_{52}'\\
y_{52} &= -0{\cdot}000\,336\ p_{12}' - 0{\cdot}000\,14\ p_{22}' + 0{\cdot}000\,20\ p_{32}' + 0{\cdot}000\,008\ p_{42}' + 0{\cdot}156\,84\ p_{52}'\\
x_2 &= +0{\cdot}000\,003\ p_{12}' + 0{\cdot}000\,099\ p_{22}' - 0{\cdot}000\,0137\ p_{32}' - 0{\cdot}000\,0002\ p_{42}' - 0\ 000\,00009\ p_{52}'
\end{aligned}$$

$$\begin{aligned}
y_{13} &= -0{\cdot}041\,03\ p_{13}' - 0{\cdot}003\,39\ p_{23}' + 0{\cdot}000\,92\ p_{33}' + 0{\cdot}000\,47\ p_{43}' + 0{\cdot}000\,11\ p_{53}'\\
y_{23} &= -0{\cdot}005\,10\ p_{13}' - 0{\cdot}199\,10\ p_{23}' + 0{\cdot}027\,88\ p_{33}' + 0{\cdot}003\,01\ p_{43}' + 0\ 000\,77\ p_{53}'\\
y_{43} &= -0{\cdot}000\,09\ p_{13}' + 0{\cdot}010\,26\ p_{23}' + 0{\cdot}200\,90\ p_{33}' + 0{\cdot}996\,78\ p_{43}' - 0\ 042\,76\ p_{53}'\\
y_{53} &= -0{\cdot}000\,13\ p_{13}' - 0{\cdot}001\,06\ p_{23}' - 0{\cdot}002\,67\ p_{33}' + 0{\cdot}000\,63\ p_{43}' + 0{\cdot}737\,21\ p_{53}'\\
x_3 &= +0{\cdot}000\,0007\ p_{13}' + 0{\cdot}000\,0039\ p_{23}' + 0{\cdot}000\,097\ p_{33}' - 0{\cdot}000\,01115\ p_{43}' - 0{\cdot}000\,0010\ p_{53}'
\end{aligned}$$

$$
\begin{array}{llllll}
y_{14} = -0{\cdot}03927 & p_{14}^{\,I} - 0{\cdot}0028 & p_{24}^{\,I} - 0{\cdot}0080 & p_{34}^{\,I} + 0{\cdot}0008 & p_{44}^{\,I} + 0{\cdot}00066 & p_{54}^{\,I} \\
y_{24} = -0{\cdot}00350 & p_{14}^{\,I} - 0{\cdot}17047 & p_{24}^{\,I} - 0{\cdot}11322 & p_{34}^{\,I} + 0{\cdot}01598 & p_{44}^{\,I} + 0{\cdot}00871 & p_{54}^{\,I} \\
y_{34} = -0{\cdot}00196 & p_{14}^{\,I} - 0{\cdot}03334 & p_{24}^{\,I} - 0{\cdot}99217 & p_{34}^{\,I} + 0{\cdot}11358 & p_{44}^{\,I} + 0{\cdot}04260 & p_{54}^{\,I} \\
y_{54} = -0{\cdot}00372 & p_{14}^{\,I} - 0{\cdot}00353 & p_{24}^{\,I} - 0{\cdot}01577 & p_{34}^{\,I} - 0{\cdot}00566 & p_{44}^{\,I} + 2{\cdot}88554 & p_{54}^{\,I} \\
x_4 = -0{\cdot}00000006 & p_{14}^{\,I} + 0{\cdot}0000008 & p_{24}^{\,I} + 0{\cdot}0000198 & p_{34}^{\,I} + 0{\cdot}0000978 & p_{44}^{\,I} - 0{\cdot}0000164 & p_{54}^{\,I}
\end{array}
$$

$$
\begin{array}{llllll}
y_{15} = -0{\cdot}03868 & p_{15}^{\,I} - 0{\cdot}00274 & p_{25}^{\,I} - 0{\cdot}00640 & p_{35}^{\,I} - 0{\cdot}00327 & p_{45}^{\,I} + 0{\cdot}00073 & p_{55}^{\,I} \\
y_{25} = -0{\cdot}00327 & p_{15}^{\,I} - 0{\cdot}15997 & p_{25}^{\,I} - 0{\cdot}08673 & p_{35}^{\,I} - 0{\cdot}03851 & p_{45}^{\,I} + 0{\cdot}00943 & p_{55}^{\,I} \\
y_{35} = +0{\cdot}00347 & p_{15}^{\,I} - 0{\cdot}02280 & p_{25}^{\,I} - 0{\cdot}77931 & p_{35}^{\,I} - 0{\cdot}23238 & p_{45}^{\,I} + 0{\cdot}05925 & p_{55}^{\,I} \\
y_{45} = +0{\cdot}03983 & p_{15}^{\,I} - 0{\cdot}01439 & p_{25}^{\,I} - 0{\cdot}40724 & p_{35}^{\,I} - 2{\cdot}78183 & p_{45}^{\,I} + 0{\cdot}45379 & p_{55}^{\,I} \\
x_5 = -0{\cdot}00000016 & p_{15}^{\,I} - 0{\cdot}00000011 & p_{25}^{\,I} - 0{\cdot}00000042 & p_{35}^{\,I} - 0{\cdot}00000024 & p_{45}^{\,I} + 0{\cdot}0001000 & p_{55}^{\,I}
\end{array}
$$

where it is to be observed that $p_{ji}^{\,I} = 10^4 . p_{ji}$

The formulae for $p_{ji}^{\,I}$ are long and complicated. We find for instance the following system:

$$
\begin{aligned}
p_{11}^{\,I} = {} & 50{\cdot}1777\,\mu_0 + 0{\cdot}14234\,\mu_5 + 0{\cdot}06548\,\nu_1 + 0{\cdot}96479\,\nu_2 + 0{\cdot}41388\,\nu_3 + 0{\cdot}03345\,\nu_4 + 0{\cdot}06530\,\nu_1\mu_0 \\
& - 0{\cdot}06515\,\mu_0\eta + 0{\cdot}00015\,\nu_2\eta + 0{\cdot}06515\,\eta^2 + 0{\cdot}06530\,\nu_1\eta^2 - 0{\cdot}06515\,\eta^3 - 0{\cdot}06533\,\eta - 0{\cdot}06548\,\nu_1\eta \\
& + 0{\cdot}00018\,\nu_1\mu_5 - 0{\cdot}00018\,\mu_5\eta + 0{\cdot}00054\,\nu_2^2 + 0{\cdot}00209\,\nu_3^2 .
\end{aligned}
$$

$$
\begin{aligned}
p_{21}^{\,I} = {} & -0{\cdot}9314\,\mu_0 - 0{\cdot}0039\,\mu_0\nu_2 - 0{\cdot}00247\,\mu_5 - 0{\cdot}0350\,\eta - 0{\cdot}0357\,\nu_1\eta - 0{\cdot}0350\,\mu_0\eta + 0{\cdot}0350\,\eta^2 \\
& + 0{\cdot}0357\,\nu_1\eta^2 - 0{\cdot}0350\,\eta^3 + 0{\cdot}0005\,\nu_2\eta - 0{\cdot}6505\,\nu_1 + 0{\cdot}0102\,\nu_2 + 0{\cdot}0281\,\nu_3 + 0{\cdot}0007\,\nu_4 \\
& + 0{\cdot}0357\,\nu_1\mu_0 - 0{\cdot}0008\,\nu_1\nu_2 - 0{\cdot}0013\,\nu_2\nu_3 + 0{\cdot}0090\,\nu_3^2 + 0{\cdot}00018\,\nu_3\eta - 0{\cdot}00062\,\nu_2^2 .
\end{aligned}
$$

$$
\begin{aligned}
p_{31}^{\,I} = {} & +0{\cdot}00288\,\mu_0 + 0{\cdot}04132\,\nu_1\mu_0 - 0{\cdot}03469\,\eta - 0{\cdot}04132\,\nu_1\eta - 0{\cdot}00615\,\nu_3\eta - 0{\cdot}04173\,\mu_0\eta + 0{\cdot}03469\,\eta^2 \\
& + 0{\cdot}04132\,\nu_1\eta^2 - 0{\cdot}03469\,\eta^3 - 0{\cdot}03625\,\nu_1 + 0{\cdot}00596\,\nu_2 + 0{\cdot}3206\,\nu_3 - 0{\cdot}00043\,\nu_4 + 0{\cdot}00026\,\nu_0 \\
& - 0{\cdot}00274\,\nu_3^2 - 0{\cdot}00123\,\nu_1\nu_3 - 0{\cdot}00864\,\nu_2\nu_3 - 0{\cdot}00190\,\nu_3 + 0{\cdot}00043\,\nu_1\nu_3\eta - 0{\cdot}00094\,\mu_5\nu_3 \\
& - 0{\cdot}34106\,\mu_0\nu_3 .
\end{aligned}
$$

$$
\begin{aligned}
p_{41}^{\,I} = {} & +0{\cdot}08518\,\mu_0 + 0{\cdot}04753\,\mu_0\nu_1 - 0{\cdot}04781\,\eta + 0{\cdot}00049\,\nu_4\mu_0 - 0{\cdot}04753\,\nu_1\eta + 0{\cdot}00021\,\nu_2\eta \\
& - 0{\cdot}00049\,\nu_4\eta - 0{\cdot}04733\,\mu_0\eta + 0{\cdot}01915\,\mu_0\nu_4 + 0{\cdot}04781\,\eta^2 + 0{\cdot}04753\,\nu_1\eta^2 + 0{\cdot}00049\,\nu_4\eta^2 \\
& - 0{\cdot}04781\,\eta^3 + 0{\cdot}03760\,\nu_1 + 0{\cdot}00131\,\nu_2 + 0{\cdot}00339\,\nu_3 - 0{\cdot}01907\,\nu_4 + 0{\cdot}00036\,\nu_2\nu_4 + 0{\cdot}00028\,\nu_3\nu_4 \\
& + 0{\cdot}00002\,\nu_2^2 + 0{\cdot}00258\,\nu_3^2 .
\end{aligned}
$$

$$
\begin{aligned}
p_{51}^{\,I} = {} & 0{\cdot}07624\,\mu_0 + 0{\cdot}04116\,\nu_1\mu_0 + 0{\cdot}0001\,\mu_5 - 0{\cdot}04118\,\eta - 0{\cdot}04116\,\nu_1\eta + 0{\cdot}00019\,\nu_2\eta - 0{\cdot}04138\,\mu_0\eta \\
& + 0{\cdot}04118\,\eta^2 + 0{\cdot}04116\,\nu_1\eta^2 - 0{\cdot}04118\,\eta^3 + 0{\cdot}00047\,\nu_2 + 0{\cdot}04116\,\nu_1 + 0{\cdot}00028\,\nu_3 .
\end{aligned}
$$

It appears from these expressions that $p_{ji}^{\,I}$ are small quantities. In only a few exceptional cases the $p_{ji}^{\,I}$ contain coefficients exceeding unity. The largest term is $50{\cdot}1777\,\mu_0$ in $p_{11}^{\,I}$, further $9{\cdot}9144\,\mu_0$ in $p_{22}^{\,I}$, $1{\cdot}93433\,\mu_0$ in $p_{33}^{\,I}$; the other terms have all coefficients less than unity, most of them are very small. Substituting these expressions in y_{ji} and x_i it appeared that it was actually allowable to neglect quantities x_i^2, y_{ji}^2, etc., for the errors arising thereby do not exceed those caused by the omitting in the developments of all corrections smaller than 0·01.

Moreover it appears that the equations are very well suited for the computation of the important corrections Δb_i, since the expressions are very small. In the numerical computation as a rule everything has been omitted that does not affect the ninth decimal place in the Δb_i and the fourth decimal place in the $\Delta\sigma_{ji}$.

The results thus derived are:

$$\Delta b_1 = +0.0025\,0382\,\mu_0 + 0.0000\,07122\,\mu_5 + 0.0000\,04551\,\nu_1 + 0.0000\,48925\,\nu_2 + 0.0000\,2094\,\nu_3 + 0.0000\,01670\,\nu_4 + 0.0000\,03219\,\nu_1\mu_0 - 0.0000\,03205\,\mu_0\eta + 0.0000\,03201\,\eta^2 + 0.0000\,03219\,\nu_1\eta^2 - 0.0000\,03201\,\eta^3 - 0.0000\,03219\,\eta - 0.0000\,03236\,\nu_1\eta + 0.0000\,00009\,\nu_1\mu_5 - 0.0000\,00009\,\mu_5\eta + 0.0000\,00055\,\nu_2^2 + 0.0000\,00193\,\nu_3^2 + 0.0000\,00014\,\nu_2\eta.$$

$$\Delta\sigma_{21} = +0.03505\,\mu_0 + 0.00011\,\mu_5 - 0.03316\,\nu_1 + 0.00077\,\nu_2 + 0.00167\,\nu_3 + 0.00009\,\nu_4 + 0.00188\,\nu_1\mu_0 - 0.00185\,\mu_0\eta + 0.00186\,[\eta^2 - \eta - \eta^3] + 0.00188\,[\nu_1\eta^2 - \nu_1\eta] - 0.00020\,\nu_2\mu_0.$$

$$\Delta\sigma_{31} = 0.00207\,\mu_0 - 0.00162\,\eta - 0.00191\,\mu_0\eta + 0.00162\,[\eta^2 - \eta^3] - 0.00162\,[\nu_1\eta - \nu_1\eta^2 - \nu_1\mu_0] - 0.00127\,\nu_1 + 0.00026\,\nu_2 + 0.00005\,\nu_3 - 0.00052\,\nu_3\eta - 0.00011\,\nu_3^2 - 0.01399\,\mu_0\nu_3.$$

$$\Delta\sigma_{41} = +0.00045\,\mu_0 - 0.00190\,[\eta - \eta^2 + \eta^3] + 0.00187\,[\nu_1\mu_0 - \nu_1\eta + \nu_1\eta^2] - 0.00189\,\mu_0\eta + 0.00150\,\nu_1 + 0.00005\,\nu_2 + 0.00003\,\nu_3 - 0.00150\,\nu_4 + 0.00010\,\mu_0\nu_3 - 0.00077\,\nu_4\mu_0 - 0.00004\,[\nu_4\eta - \nu_4\eta^2] + 0.00010\,\nu_3^2.$$

$$\Delta\sigma_{51} = +0.00023\,\mu_0 + 0.00159\,[\nu_1\mu_0 + \nu_1] - 0.00005\,\nu_2 - 0.00002\,\nu_3 - 0.00158\,[\eta - \eta^2 + \eta^3 + \mu_0\eta] - 0.00159\,[\nu_1\eta - \nu_1\eta^2].$$

$$\Delta b_2 = +0.0004\,8908\,\mu_0 - 0.0000\,0337\,\nu_2 - 0.0000\,0381\,\nu_1 + 0.0000\,7910\,\nu_3 + 0.0000\,03959\,\nu_4 - 0.0000\,0424\,[\eta - \eta^2 + \eta^3] - 0.0000\,0057\,\nu_2\eta - 0.0000\,0422\,\mu_0\eta + 0.0000\,0055\,\mu_5 - 0.0000\,0389\,\nu_1\mu_0 + 0.0000\,0362\,\nu_1\eta^2 + 0.0000\,0002\,\nu_1^2\mu_0 + 0.0000\,0001\,\nu_3\eta - 0.0000\,00022\,[\nu_1^2\eta - \nu_1^2\eta^2] + 0.0000\,00012\,\nu_4^2 - 0.0000\,00180\,\nu_1^2 - 0.0000\,0365\,\nu_1\eta + 0.0000\,00024\,\nu_1\nu_3 + 0.0000\,00133\,\nu_3^2.$$

$$\Delta\sigma_{12} = +0.02129\,\nu_2 - 0.02182\,\mu_0 + 0.00023\,[\eta - \eta^2 + \eta^3] + 0.00025\,\nu_2\eta - 0.00004\,\mu_0\eta - 0.00007\,\mu_5 - 0.00008\,\nu_1 + 0.00067\,\nu_3 + 0.00003\,\nu_4 - 0.00459\,\nu_1\mu_0 + 0.0000\,\nu_1\eta^2 - 0.00001\,[\nu_1^2\mu_0 + \nu_1\mu_5 + \nu_3\eta + \nu_3^2] - 0.00009\,\nu_1^2 - 0.00007\,\nu_1\nu_2 - 0.00001\,\nu_2\eta - 0.00015\,\nu_1\nu_3.$$

$$\Delta\sigma_{22} = +0.03150\,\mu_0 - 0.00247\,\nu_1 + 0.03856\,\nu_2 + 0.00808\,\nu_3 + 0.00023\,\nu_1\mu_0 + 0.00004\,\mu_5 - 0.00070\,\nu_4 + 0.00069\,[\eta + \mu_0\eta - \eta^2 + \eta^3] - 0.00048\,\nu_1\eta + 0.00114\,\nu_2\eta - 0.00002\,\nu_1^2 - 0.00006\,\nu_2\nu_3 + 0.00017\,[\nu_4^2 + \nu_1\eta^2] - 0.00022\,\nu_1\nu_3 - 0.00026\,\nu_3^2 - 0.00002\,\nu_3\nu_4 + 0.00014\,\mu_0\nu_4 - 0.00018\,\nu_1\nu_4.$$

$$\Delta\sigma_{42} = -0.00226\,\nu_2 + 0.00015\,\nu_1 + 0.00042\,\mu_0 + 0.00117\,\nu_3 + 0.00003\,\nu_4 - 0.00126\,[\eta - \eta^2 + \eta^3] - 0.00133\,\mu_0\eta - 0.00116\,\nu_2\eta - 0.00019\,\nu_1\eta + 0.00003\,\nu_3\eta + 0.00002\,\nu_1^2 + 0.00005\,\nu_3^2 - 0.00005\,\nu_4^2 - 0.00133\,\nu_3\nu_4 + 0.00019\,[\nu_1\mu_0 + \nu_1\eta^2] - 0.00676\,\nu_4\mu_0 + 0.00002\,[\nu_1^2\mu_0 - \nu_1^2\eta + \nu_1^2\eta^2] - 0.0009\,\nu_4\eta^2 - 0.0008\,\nu_1\nu_4 - 0.00002\,\nu_2\nu_4.$$

$$\Delta\sigma_{52} = -0.00072\,\nu_2 + 0.00013\,\mu_0 + 0.00010\,\nu_1 - 0.00014\,\nu_3 + 0.00012\,\nu_1\mu_0 - 0.00091\,\nu_2\eta - 0.00101\,\eta + 0.00101\,\eta^2 - 0.00015\,[\nu_1\eta - \nu_1\eta^2] + 0.00003\,\nu_3\eta - 0.00139\,\mu_0\eta - 0.00100\,\eta^3 + 0.00002\,[\nu_1^2\eta^2 - \nu_1^2\eta + \nu_1^2 + \nu_1^2\mu_0].$$

$$\Delta b_3 = +0.0000\,96712\,\mu_0 - 0.0000\,00040\,\mu_5 + 0.0000\,07640\,\nu_1 + 0.0000\,15235\,\nu_2 - 0.0000\,24442\,\nu_3 - 0.0000\,40384\,\nu_4 - 0.0000\,00401\,\eta + 0.0000\,00441\,\nu_3\eta - 0.0000\,00426\,\mu_0\eta + 0.0000\,00404\,[\eta^2 - \eta^3] + 0.0000\,000365\,\nu_1\mu_0 - 0.0000\,00029\,[\nu_1\eta - \nu_1^2\eta] - 0.0000\,00089\,\nu_2\eta - 0.0000\,00849\,[\nu_4\eta^2 + \nu_4\mu_0].$$

$$\Delta\sigma_{13} = -0.00029\,\nu_1 + 0.00232\,\nu_2 + 0.00867\,\nu_3 - 0.01060\,\mu_0 + 0.00018\,\eta + 0.00001\,[\nu_1\eta + \nu_2\eta] + 0.00014\,\nu_3\eta + 0.00129\,\mu_0\eta - 0.00018\,[\eta^2 - \eta^3] - 0.00002\,\mu_5.$$

$$\Delta\sigma_{23} = -0\cdot01475\,\nu_1 + 0\cdot0098\,\nu_2 + 0\cdot13459\,\nu_3 + 0\cdot0019\,\nu_4 - 0\cdot1097\,\mu_0 - 0\cdot0001\,\nu_1\varkappa + 0\cdot0008\,\varkappa + 0\cdot0006\,\nu_3\varkappa - 0\cdot0008\,[\varkappa^2 - \varkappa^3] - 0\cdot0001\,\nu_1\varkappa^2 + 0\cdot00093\,\mu_0\varkappa - 0\cdot00017\,\nu_1\mu_0 - 0\cdot0002\,\nu_4\varkappa^2.$$

$$\Delta\sigma_{43} = +0\cdot0275\,\nu_2 - 0\cdot2386\,\nu_3 + 0\cdot1794\,\mu_0 - 0\cdot1122\,\nu_1 + 0\cdot0304\,\nu_4 + 0\cdot0001\,\mu_5 - 0\cdot00435\,\varkappa - 0\cdot0003\,\nu_1\varkappa - 0\cdot0055\,\nu_3\varkappa - 0\cdot0052\,[\mu_0\varkappa - \varkappa^2 + \varkappa^3] + 0\cdot0003\,[\nu_1\mu_0 + \nu_1\varkappa^2] - 0\cdot0009\,\nu_2\varkappa - 0\cdot0002\,[\nu_4\varkappa^2 + \nu_4\mu_0].$$

$$\Delta\sigma_{53} = -0\cdot00005\,\nu_1 + 0\cdot00017\,\nu_2 + 0\cdot00364\,\nu_3 + 0\cdot0006\,\mu_0 - 0\cdot0002\,\nu_1\varkappa - 0\cdot0039\,[\mu_0\varkappa + \varkappa - \varkappa^2 + \varkappa^3] \; 0\cdot000023\,[\nu_4\varkappa^2 + \nu_4\mu_0] + 0\cdot00025\,[\nu_1\mu_0 + \nu_1\varkappa^2] - 0\cdot0006\,\nu_2\varkappa - 0\cdot0030\,\nu_3\varkappa - 0\cdot0004\,\nu_4$$

$$\Delta b_4 = +0\cdot0000\,15418\,\mu_0 - 0\cdot0000\,00001\,\mu_5 + 0\cdot0000\,00817\,\nu_1 + 0\cdot0000\,01583\,\nu_2 + 0\cdot0000\,14944\,\nu_3 - 0\cdot0000\,01976\,\nu_4 + 0\cdot0000\,00051\,\nu_4\mu_0 - 0\cdot0000\,00126\,\varkappa - 0\cdot0000\,00051\,[\nu_4\varkappa - \nu_4\varkappa^2] + 0\cdot0000\,00109\,\varkappa^2 - 0\cdot0000\,00092\,\varkappa^3 - 0\cdot0000\,00041\,\nu_1\varkappa - 0\cdot0000\,00140\,\mu_0\varkappa - 0\cdot0000\,00015\,\nu_1\mu_0 - 0\cdot0000\,00033\,\nu_1^2\varkappa^2 - 0\cdot0000\,00012\,\nu_2\varkappa - 0\cdot0000\,00081\,\nu_1^2 - 0\cdot0000\,00063\,\nu_3\varkappa - 0\cdot0000\,00038\,\nu_1\varkappa^2 + 0\cdot0000\,00017\,\nu_1^2\varkappa.$$

$$\Delta\sigma_{14} = -0\cdot00008\,\nu_1 + 0\cdot00014\,\nu_2 + 0\cdot00096\,\nu_3 + 0\cdot0013\,\nu_4 - 0\cdot0025\,\mu_0 + 0\cdot0025\,\nu_1\mu_0 - 0\cdot00006\,\nu_2\varkappa - 0\cdot0032\,\mu_0\varkappa.$$

$$\Delta\sigma_{24} = -0\cdot00242\,\nu_1 - 0\cdot00177\,\nu_2 + 0\cdot0134\,\nu_3 + 0\cdot01598\,\nu_4 - 0\cdot02603\,\mu_0 + 0\cdot00033\,\varkappa + 0\cdot0011\,\mu_0\varkappa - 0\cdot0003\,\varkappa^2 + 0\cdot0003\,\varkappa^3 + 0\cdot0004\,\nu_1\varkappa - 0\cdot0002\,\nu_1\mu_0 + 0\cdot0001\,\nu_3\varkappa - 0\cdot0003\,\nu_1^2\varkappa^2 - 0\cdot0003\,[\nu_1^2 - \nu_1^2\varkappa + \nu_1^2\mu_0] - 0\cdot0004\,\nu_1\varkappa^2 - 0\cdot00015\,[\nu_4\mu_0 - \nu_4\varkappa + \nu_4\varkappa^2].$$

$$\Delta\sigma_{34} = -0\cdot0065\,\nu_1 - 0\cdot0158\,\nu_2 + 0\cdot0193\,\nu_3 + 0\cdot0966\,\nu_4 - 0\cdot0975\,\mu_0 + 0\cdot0009\,\varkappa + 0\cdot0004\,\nu_1\varkappa - 0\cdot0008\,\varkappa^2 + 0\cdot0007\,\varkappa^3 + 0\cdot0005\,\mu_0\varkappa - 0\cdot0003\,\nu_1\mu_0 + 0\cdot0003\,\nu_1^2\varkappa + 0\cdot0004\,\nu_3\varkappa + 0\cdot0007\,[\nu_4\varkappa - \nu_4\varkappa^2] - 0\cdot00072\,\nu_4\mu_0 - 0\cdot0009\,\nu_1\varkappa^2 - 0\cdot0001\,\nu_1^2\varkappa^2 - 0\cdot0001\,\nu_1^2.$$

$$\Delta\sigma_{54} = -0\cdot0001\,\nu_1 - 0\cdot0005\,\nu_3 + 0\cdot0027\,\nu_4 - 0\cdot0015\,\mu_0 - 0\cdot0015\,\varkappa + 0\cdot0006\,\nu_1\varkappa + 0\cdot0010\,\varkappa^2 - 0\cdot0007\,\varkappa^3 - 0\cdot0010\,\mu_0\varkappa - 0\cdot0014\,\nu_1\mu_0 + 0\cdot0011\,[\nu_4\mu_0 - \nu_4\varkappa - \nu_1^2\varkappa^2 + \nu_4\varkappa^2] - 0\cdot0003\,\nu_2\varkappa + 0\cdot0007\,\nu_1^2\varkappa - 0\cdot0013\,\nu_1\varkappa^2.$$

$$\Delta b_5 = -0\cdot0000\,0001\,7321\,\mu_0 - 0\cdot0000\,0003\,5953\,\varkappa + 0\cdot0000\,0000\,0653\,\varkappa^2 + 0\cdot0000\,0003\,1941\,\mu_0\varkappa + 0\cdot0000\,0003\,4513\,\varkappa^3 - 0\cdot0000\,0003\,5087\,\nu_1\mu_0 - 0\cdot0000\,0013\,9689\,\nu_1\varkappa^2 + 0\cdot0000\,0006\,9637\,\nu_1\varkappa + 0\cdot0000\,0000\,0098\,[\nu_1\mu_5 + \mu_0\mu_5 - \mu_5\varkappa - \nu_1\mu_5\varkappa + \mu_5\varkappa^2 + \nu_1\mu_5\varkappa^2] - 0\cdot0000\,0000\,3509\,\nu_2\varkappa + 0\cdot0000\,0000\,0283\,[\nu_1\nu_3 - \nu_1\nu_3\varkappa] - 0\cdot0000\,0001\,2090\,\nu_3\varkappa + 0\cdot0000\,0000\,5827\,[\nu_4\mu_0 - \nu_4\varkappa + \nu_4\varkappa^2] - 0\cdot0000\,0000\,00144\,\mu_5 + 0\cdot0000\,0000\,0835\,\nu_1 + 0\cdot0000\,0000\,1093\,\nu_2 + 0\cdot0000\,0000\,8879\,\nu_3 + 0\cdot0000\,0001\,1506\,\nu_4 - 0\cdot0000\,0003\,5997\,\nu_1^2 - 0\cdot0000\,0007\,1062\,\nu_1^2\mu_0 + 0\cdot0000\,0007\,1242\,\nu_1^2\varkappa - 0\cdot0000\,0010\,7281\,\nu_1^2\varkappa^2 + 0\cdot0000\,0000\,0661\,[\nu_1\nu_2 - \nu_1\nu_2\varkappa] + 0\cdot0000\,0000\,0023\,[\nu_1\nu_4 - \nu_1\nu_4\varkappa] - 0\cdot0000\,0000\,0080\,\nu_2^2 - 0\cdot0000\,0000\,0008\,\nu_3^2 + 0\cdot0000\,0000\,0010\,\nu_2\nu_3 + 0\cdot0000\,0000\,00005\,\nu_2\nu_4 + 0\cdot0000\,0000\,0000\,8\,\nu_3\nu_4.$$

$$\Delta\sigma_{15} = -0\cdot00002\,\nu_1 - 0\cdot00889\,\mu_0 + 0\cdot00833\,\nu_1\mu_0 + 0\cdot00002\,\varkappa - 0\cdot00060\,\mu_0\varkappa - 0\cdot00001\,\varkappa^2 + 0\cdot00015\,\nu_4 + 0\cdot00002\,[\nu_1\mu_5 - \mu_5\varkappa] + 0\cdot00016\,\nu_1\nu_2 - 0\cdot00015\,\nu_2\varkappa - 0\cdot00016\,\nu_1\nu_2\varkappa + 0\cdot00007\,[\nu_1\nu_3 - \nu_1\nu_3\varkappa] - 0\cdot00006\,\nu_3\varkappa.$$

$$\Delta\sigma_{25} = -0\cdot00829\,\mu_0 - 0\cdot00044\,\nu_2 + 0\cdot00025\,\varkappa - 0\cdot00016\,\varkappa^2 + 0\cdot00006\,\varkappa^3 - 0\cdot00071\,\nu_1 + 0\cdot00162\,\nu_3 + 0\cdot00049\,\nu_4 - 0\cdot00023\,\nu_1\mu_0 + 0\cdot00583\,\nu_1\varkappa^2 + 0\cdot00095\,\nu_1\varkappa + 0\cdot000024\,[\nu_1\nu_2 - \nu_1\nu_2\varkappa] - 0\cdot00088\,\nu_1^2 - 0.0008\,[\nu_1^2\mu_0 - \nu_1^2\varkappa] - 0\cdot00070\,\nu_1^2\varkappa^2 + 0\cdot000015\,[\nu_4\varkappa - \nu_4\varkappa^2]$$

$$\Delta\sigma_{35} = -0{\cdot}0276\,\mu_0 + 0{\cdot}0015\,\eta + 0{\cdot}0011\,\eta^2 + 0{\cdot}0007\,\eta^3 + 0{\cdot}0007\,\mu_0\eta - 0{\cdot}0015\,\nu_1 - 0{\cdot}0049\,\nu_3 - 0{\cdot}0018\,\nu_1\mu_0 - 0{\cdot}0003\,\nu_1\eta^2 + 0{\cdot}0011\,\nu_1\eta - 0{\cdot}0003\,\nu_1^2 + 0{\cdot}0001\,\nu_2\eta + 0{\cdot}0001\,\nu_1^2\mu_0 - 0{\cdot}0001\,\nu_1^2\eta - 0{\cdot}0004\,\nu_1^2\eta^2 + 0{\cdot}0108\,\nu_4 + 0{\cdot}0001\,\nu_3\eta - 0{\cdot}0001\,[\nu_4\mu_0 - \nu_4\eta + \nu_4\eta^2] - 0.0038\,\nu_2$$

$$\Delta\sigma_{45} = -0{\cdot}0716\,\mu_0 + 0{\cdot}0095\,\eta - 0{\cdot}0881\,\eta^2 + 0{\cdot}0072\,\eta^3 + 0{\cdot}0079\,\mu_0\eta - 0{\cdot}0115\,\nu_1 - 0\,0073\,\nu_2 - 0{\cdot}0620\,\nu_3 + 0{\cdot}0054\,\nu_4 - 0{\cdot}0153\,\nu_1\mu_0 - 0{\cdot}0045\,\nu_1\eta^2 + 0{\cdot}0076\,\nu_1\eta - 0{\cdot}0002\,[\nu_1\nu_2 - \nu_1\nu_2\eta] + 0{\cdot}0003\,\nu_2\eta + 0{\cdot}0004\,\nu_3\eta + 0{\cdot}0005\,\nu_1^2 + 0{\cdot}0016\,[\nu_1^2\mu_0 - \nu_1^2\eta] + 0{\cdot}0028\,\nu_1^2\eta^2 - 0{\cdot}0002\,[\nu_4\mu_0 - \nu_4\eta + \nu_4\eta^2].$$

These developments show, that the values of the b_j and σ_{ij} are very uncertain, since the corrections μ_0, ν_i, etc. are themselves relatively large Moreover it appears that the motions of the nodes chiefly depend on the compression of the planet; the influence of the higher compression terms is only perceptible for the first satellite; in the case of the other satellites it is practically negligible.

CHAPTER V.

COMPARISON WITH SOUILLART'S THEORY.

A. Comparison of the perturbative functions.

In the derivation of the differential equations of a satellite Souillart has taken into account the compression of Jupiter and for the potential function of ♃ with respect to a satellite he takes: $f\frac{M}{r} + fV$, where f is the Gaussian constant of attraction, M the mass of Jupiter and r the distance of the satellite from the centre of gravity, the perturbative function V in his theory being: $V = \frac{J_1}{r^3}\left(\frac{1}{3} - \sin^2 d\right)$, where J_1 is the mass of ♃ multiplied by the square of the equatorial radius b and by $e_1 - \frac{1}{2}\varphi$, e_1 being the ellipticity of the free surface, φ the ratio of the centrifugal force and gravity at the equator of ♃, and d the jovicentric latitude of a satellite.

Other terms of the perturbative function have been neglected by Souillart. From the developments of the b_i and σ_{ij} in the preceding chapter, we can make an estimate of the influence of the several parts of the perturbative function on the values of the b_i and σ_{ij}. We find that if the b_i and σ_{ij} are computed in as many decimal places as has heen done by Souillart, the higher compression-terms of the perturbative function cannot entirely be neglected. If we disregard the latter the perturbative functions arising from the figure of the planet agree in both methods, since: $J_1 = Mb^2\left(e_1 - \frac{\varphi}{2}\right) = \frac{3}{2}(C-A)$ and the corresponding compression-term used by us is: $MJb^2 = \frac{3}{2}(C-A)$.

Souillart's equations of motion are referred to a fixed plane of (x, y), for which, in the numerical evaluation of his theory he has chosen the orbital plane at the epoch 1850·0. In order to derive these equations he makes use of elliptical elements. The equations then are:

$$\frac{d\theta}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin\varphi}\frac{\partial R}{\partial \varphi}$$

$$\frac{d\varphi}{dt} = -\frac{1}{na^2\sqrt{1-e^2}\sin\varphi}\frac{\partial R}{\partial \theta} - \frac{\tan\frac{1}{2}\varphi}{na^2\sqrt{1-e^2}}\left(\frac{\partial R}{\partial \varpi} + \frac{\partial R}{\partial \varepsilon}\right)$$

In the right hand members of these equations Souillart has neglected all terms of the third order, regarding φ, e, and m (the mass of a satellite expressed in the mass of Jupiter as unit) as quantities of the first order of magnitude. With this approximation the equations become:

$$(53) \qquad \frac{d\theta}{dt} = \frac{1}{na^2\varphi}\frac{\partial R}{\partial \varphi} \qquad \frac{d\varphi}{dt} = -\frac{1}{na^2\varphi}\frac{\partial R}{\partial \theta}$$

which form the starting point of his theory of the inclinations and nodes.

The perturbative function R is composed of several parts. We have:

$$R = \Sigma_j\, fm_j\, R_{ji} + fm_s\, R_{si} + fV_i,\ j = 1, \ldots . 4.$$

where R_{ji} is the perturbative function caused by the satellite j on i, R_{si} the part due to the Sun and V_i the part of V, that relates to satellite i.

The development of the perturbative function used by Souillart is that of Leverrier, as regards the parts arising from the satellites; for R_{si} he has employed another development, since the ratio $\frac{r_i}{r_s}$ is very small, smaller than $\frac{1}{400}$, so that one term of the development was sufficient for his approximation.

Denoting the secular terms of the second order in the inclinations by S [ ], we have from Souillart's theory:

$$\mathrm{S}\left[R_{si}\right] = -\frac{3}{8}\frac{a_i^2}{a_s^3}\left[\varphi_i^2 + \varphi_s^2 - 2\,\varphi_i\,\varphi_s\,\cos(\theta_i - \theta_s)\right]$$

$$\mathrm{S}\left[R_{ji}\right] = -\frac{1}{8}\,B_{ji}^{(1)}\left[\varphi_i^2 + \varphi_j^2 - 2\,\varphi_i\,\varphi_j\cos(\theta_i - \theta_j)\right]$$

$$\mathrm{S}\left[V_i\right] = \frac{f_1}{a_i^3}\left[-\frac{1}{2}\varphi_i^2 - \varphi_i\,\pi\,\cos(\theta_i + \psi)\right]$$

where $B_{ji}^{(1)} = \frac{a_i}{a_j^2}\,c^{(i)}$ $(a_i < a_j)$, and $c^{(i)}$ is a Laplacean coefficient in the development:

$$(1 + \alpha^2 - 2\,\alpha\,\cos\psi)^{-\frac{3}{2}} = \frac{1}{2}\Sigma\, c^{(i)}\cos i\,\psi$$

φ_i is the inclination of a satellite i, θ_i the longitude of the node, a_i the semi-axis maior, ψ is the longitude of the descending node of Jupiter's true equator and π its inclination to the fixed plane of (x, y).

The differential equations of precession and nutation, as used by Souillart, are:

$$(54) \qquad \frac{d\pi}{dt} = \frac{1}{Cn\,\sin\pi}\frac{\partial U}{\partial \psi}, \qquad \frac{d\psi}{dt} = -\frac{1}{Cn\,\sin\pi}\frac{\partial U}{\partial \pi}$$

where n is the velocity of rotation of the planet round its axis and U the perturbative function.

We then have:

$$U = -\frac{3}{4}\,\Sigma_i\, f\frac{m_i}{a_i^3}\,(2\,C - A - B)\,\frac{a_i^3}{r_i^3}\,\frac{z_i^2}{r_i^2}$$

where A, B, and C are the principal moments of inertia of $\mathcal{Y}$, z_i the distance from the satellite to $\mathcal{Y}$'s equator. The development of U in terms of inclinations and nodes is, neglecting terms of the third order:

$$S\,[U_i] = -\frac{3}{4}\frac{fm_i}{a_i^3}(2C - A - B)\left[-\tfrac{1}{2}\pi^2 - \varphi_i\pi\cos(\vartheta_i + \psi)\right]$$

Instead of using the inclinations and nodes themselves as variables, he introduces, as is customary, the variables:

$$p_i = \varphi_i \sin\vartheta_i \quad , \quad q_i = \varphi_i \cos\vartheta_i \quad , \quad \text{where } i = 1, \ldots 4$$
$$\text{and: } p_s = \pi \sin\psi \quad , \quad q_s = -\pi\cos\psi .$$

[It may be remembered that our variables are *with the same approximation*:

$p_i = \varphi_i \cos\vartheta_i$, $q_i = -\varphi_i \sin\vartheta_i$, $i = 1, \ldots 4$, omitting the factor $\sqrt{G_i}$,
and: $p_s = \pi\cos\psi$, $q_s = \pi\sin\psi$

and thus the values of b_i found by us must have the opposite sign to those of Souillart.]

The differential equations (53) and (54) then become:

$$\frac{dp_i}{dt} = \frac{1}{n_i a_i^2}\frac{\partial R}{dq_i} \quad , \quad \frac{dq_i}{dt} = -\frac{1}{n_i a_i^2}\frac{\partial R}{\partial p_i},$$

where i takes the values 1, ... 4, and:

$$\frac{dp_s}{dt} = \frac{\cos\pi}{Cn}\frac{\partial U}{dq_5} \quad , \quad \frac{dq_s}{dt} = -\frac{\cos\pi}{Cn}\frac{\partial U}{\partial p_5}$$

Now the secular parts of the perturbative functions are, according to Souillart:

$$S\left[R_{si}\right] = -\frac{3}{8}\frac{m_s a_i^2}{a_s^3}\left[p_i^2 + q_i^2 + p_s^2 + q_s^2 - 2q_iq_s - 2p_ip_s\right]$$

$$S\left[R_{ji}\right] = -\frac{1}{8}m_j B_{ji}^{(1)}\left[p_i^2 + q_i^2 + p_j^2 + q_j^2 - 2p_ip_j - 2q_iq_j\right]$$

$$S\left[V_i\right] = -\frac{J_1}{2a_i^3}\left[p_i^2 + q_i^2 + p_s^2 + q_s^2 - 2p_ip_s - 2q_iq_s\right]$$

$$S\left[U_i\right] = -\frac{3}{8}\frac{fm_i}{a_i^3}(2C - A - B)\left[p_s^2 + q_s^2 + p_i^2 + q_i^2 - 2p_ip_s - 2q_iq_s\right]$$

The parts $S\,[R_{si}]$, $S\,[R_{ji}]$ arise from the reciprocals of the distances $\frac{1}{\Delta_{si}}$ and $\frac{1}{\Delta_{ji}}$ between the Sun or the satellite j and the satellite i. They are entirely independent of the choice of the plane of (x,y), whence $S\,[R_{si}]$ and $S\,[R_{ji}]$ according to the two methods can directly be compared with one another; the forms of the developments must be the same, even where the meaning of p_i and q_i is different.

On the other hand the developments of $S[V_i]$ and $S[U_i]$ cannot be directly compared. The plane of (x,y) as used by Souillart does not coincide with $\mathcal{Y}$'s true equator, and thus the jovicentric latitude s_i of a satellite is defined by:

(55) $$\sin s_i = \varphi_i \sin(v_i - \vartheta_i) + \pi\sin(v_i + \psi)$$

where v_i is the true longitude in the orbit, while we could put:

$$\sin s_i = \gamma_i \sin(v_i - \vartheta_i)$$

and since in both theories quantities of third order in the inclinations have been neglected, it is allowable to put:

$$\varphi_i = \gamma_i$$

Inserting (55) in our $S[U_i]$ and $S[V_i]$ it immediately appears that they agree with Souillart's expressions, since we have, neglecting as Souillart had done the higher compression-terms, for U_i and V_i:

$$fm_iM\left[-\frac{Jb^2}{r_i^3}\sin^2 s_i\right] \text{ and } -fM\left[\frac{Jb^2}{r_i^3}\sin^2 s_i\right]$$

whilst Souillart has:

$$-\frac{3}{4}f\frac{m_i}{r_i^3}(2C-A-B)\sin^2 s_i \quad \text{and} \quad -f\frac{J_1}{r_i^3}\sin^2 s_i$$

but as has been shown in the preceding: $MJb^2 = J_1$ and if we regard the figure of $\mathfrak{L}$ as an ellipsoid of revolution: $-\frac{3}{4}f\frac{m_i}{r_i^3}(2C-A-B)\sin^2 s_i = -f\frac{J_1 m_i}{r_i^3}$, since: $J_1 = \frac{3}{2}(C-A)$.

In our development of the perturbative functions we have made no difference between the parts depending on the satellites and that depending on the Sun; we therefore have to compare one and the same development with both developments of Souillart. We begin with the comparison of the parts depending on the satellites.

We found:

$$S[R_{ji}] = \Sigma_j \frac{m_j}{a'} d_{ji}^{(1)}\left[-\frac{1}{4}(p_i^2+q_i^2+p_j^2+q_j^2-2p_ip_j-2q_iq_j)\right] \tag{56}$$

where:

$$d_{ji}^{(1)} = 2b^{(2)} + 4b^{(4)} + 6b^{(6)} + \ldots\ldots \quad \text{and } a' \text{ is the larger of } a_i \text{ and } a_j$$

The corresponding value of Souillart is:

$$S[R_{ji}] = -\frac{1}{8}\Sigma_j m_j B_{ji}^{(1)}[p_i^2+q_i^2+p_j^2+q_j^2-2p_ip_j-2q_iq_j] \tag{57}$$

The expressions (56) and (57) are identical if:

$$\frac{1}{2}B_{ji}^{(1)} = \frac{d_{ji}^{(1)}}{a'} \tag{58}$$

In order to prove this relation we start from the theorem; If a function of one or more variables has been developed in more than one manner in terms of ascending powers of one of the variables and the series are convergent in the same finite interval, then the coefficients of the same powers in the different developments are the same. — Now we can show that the developments in terms of inclinations according to Leverrier and Hill are convergent at the same time; also that the terms of the second degree in the inclinations used by Souillart and by us are the only possible ones; it is then legitimate to conclude that their coefficients must be the same.

$S[R_{ji}]$ is derived from the development of $\frac{1}{\Delta_{ji}}$. Leverrier has not developed $\frac{1}{\Delta_{ji}}$ in terms of the inclinations with respect to a fixed plane of (x,y), but in terms of mutual inclinations. He finds: *)

$$\frac{1}{\Delta_{ji}} = \frac{1}{\left[r_i^2+r_j^2-2r_ir_j\cos V_{ij}\right]^{\frac{1}{2}}}$$

$$= \frac{1}{\left\{r_i^2+r_j^2-2r_ir_j\cos(v_i-v_j+\tau_j-\tau_i)\right\}^{\frac{1}{2}}}\left[1+\frac{4\varkappa_{ji}^2 r_ir_j\sin(v_i-\tau_i)\sin(v_i-\tau_j)}{r_i^2+r_j^2-2r_ir_j\cos(v_i-v_j+\tau_j-\tau_i)}\right]^{-\frac{1}{2}}$$

*) Cf. Leverrier, Annales de l'Obs. de Paris, Tome 1, p. 258.

where v_i is the true longitude, $\varkappa_{ji} = \sin\frac{I}{2}$ (I is the mutual inclination of the orbits of the two satellites) and τ_i and τ_j depend on the inclination and node of a satellite's orbit with respect to the plane of (x, y) by the formula:

$$\tau_i - \tau_j = 2 \tan\frac{\varphi_i}{2} \tan\frac{\varphi_j}{2} \sin(\theta_i - \theta_j)$$

The form in brackets can be developed in terms of $\varkappa_{ji}^2$, if $\frac{4 r_i r_j}{(r_i - r_j)^2} \sin^2\frac{I}{2} < 1$. In the development in terms of the inclinations we can neglect the eccentricities, and thus put: $r_i = a_i$, $r_j = a_j$, whence:

$$\frac{4 r_i r_j}{(r_i - r_j)^2} \sin^2\frac{I}{2} = \frac{4\alpha}{(1-\alpha)^2} \sin^2\frac{I}{2}, \text{ where } \alpha = \frac{a_i}{a'} \tag{59}$$

In the theory of the satellites $\alpha < 0{\cdot}63$ en $\frac{I}{2} < 30'$ and thus (59) being smaller than unity, the convergency of [ ] and at the same time that of $\frac{1}{\Delta_{ij}}$ is proved. Moreover it appears from the smallness of (59), that the convergency is rapid.

The terms of the second order in the inclinations are:

$$-\tfrac{1}{2}\varkappa_{ji}^2 \Sigma_i B_{ji}^{(i-1)} \cos i\,(l_j - \lambda_{ji}) + \tfrac{1}{2}\varkappa_{ji}^2 \Sigma_i B_{ji}^{(i-1)} \cos\left[i\,l_j - (i-2)\,\lambda_{ji} - 2\tau_j\right]. \tag{60}$$

where i takes the values $+\infty$ to $-\infty$, $\lambda_{ji} = l_i + \tau_j - \tau_i$, l_i being the mean longitude of satellite i.

It immediately appears from (60), that the only secular terms arise in the case $i = 0$; they thus are:

$$-\tfrac{1}{2}\varkappa_{ji}^2 B_{ji}^{(1)}$$

If we express $\varkappa_{ji}$ in terms of φ_i and φ_j by means of the relations:

$$2\varkappa_{ji} \sin\tau_j = \varphi_i \sin\theta_i - \varphi_j \sin\theta_j$$
$$2\varkappa_{ji} \cos\tau_j = \varphi_i \cos\theta_i - \varphi_j \cos\theta_j$$

we find:

$$4\varkappa_{ji}^2 = \varphi_i^2 + \varphi_j^2 - 2\varphi_i\varphi_j \cos(\theta_i - \theta_j)$$

The only secular terms of the second order in the inclinations thus are:

$$-\tfrac{1}{8} B_{ji}^{(1)} \left(\varphi_i^2 + \varphi_j^2 - 2\varphi_i\varphi_j \cos(\theta_i - \theta_j)\right) \tag{61}$$

By another reasoning it is possible to verify that (61) contains all terms of the second order in the inclinations in Leverriers's development. For, neglecting the eccentricities and all that is of a higher degree than the second in the inclinations, we have:

$$\frac{1}{\Delta_{ji}} = \frac{1}{\left[a_i^2 + a_j^2 - 2 a_i a_j \cos V_{ji}\right]^{\frac{1}{2}}} = \left(P - a_i a_j Q\right)^{-\frac{1}{2}} \tag{62}$$

where:

$$P = a_i^2 + a_j^2 - 2\, a_i a_j \cos(m_j - m_i)$$

$$Q = \left(-2\sin^2\frac{\varphi_i}{2} - 2\sin^2\frac{\varphi_j}{2}\right)\cos(m_j - m_i) + \sin\varphi_i \sin\varphi_j \cos(m_j - m_i - \vartheta_j + \vartheta_i)$$
$$- \sin\varphi_i \sin\varphi_j \cos(m_j + m_i - \vartheta_i - \vartheta_j)$$

m_i and m_j denoting the mean longitudes of the satellites i and j.

It appears that Q is of the second degree in the inclinations.

Now we have, with the values of α and φ_i:

$$\left|\frac{a_i a_j}{P} Q\right| < 1$$

so that (62) can be developed in a converging series:

$$\left(P - a_i a_j Q\right)^{-\frac{1}{2}} = P^{-\frac{1}{2}} + \tfrac{1}{2} a_i a_j P^{-\frac{3}{2}} Q + \ldots\ldots + \frac{1.3\ldots(2k-1)}{2.4\ldots\ldots 2k} a_i^k a_j^k P^{-(k+\frac{1}{2})} Q^k + \ldots\ldots$$

The only secular terms of the second degree occurring in this expression originate from:

(63) $$\tfrac{1}{2} a_i a_j P^{-\frac{3}{2}} Q .$$

Now $a_i a_j P^{-\frac{3}{2}}$ can be developed in a converging Fourier series:

$$a_i a_j P^{-\frac{3}{2}} = \tfrac{1}{2} \sum_{-\infty}^{+\infty} B_{ji}^{(i)} \cos(m_j - m_i)$$

The secular terms are found by putting $i = 1$ in (63), thus from:

$$\tfrac{1}{2} B_{ji}^{(1)} \left[\left(-2\sin^2\frac{\varphi_i}{2} - 2\sin^2\frac{\varphi_j}{2}\right)\cos^2(m_j - m_i) + \sin\varphi_i \sin\varphi_j \cos^2(m_j - m_i)\cos(\vartheta_i - \vartheta_j)\right]$$

They are:

$$-\tfrac{1}{8} B_{ji}^{(1)} \left(\varphi_i^2 + \varphi_j^2 - 2\varphi_i\varphi_j \cos(\vartheta_i - \vartheta_j)\right)$$

which entirely agrees with the expression obtained above.

Hill's method of procedure is quite different from that of Leverrier. According to Hill $\frac{1}{\Delta_{ji}}$ is expressed in a Fourier series, the common Laplacean development:

(64) $$\frac{1}{\Delta_{ji}} = \tfrac{1}{2} B^{(0)} + B^{(1)} \cos V_{ji} + B^{(2)} \cos 2\, V_{ji} + \ldots\ldots$$

a rapidly converging series because of the values r_i, r_j in the theory of the Satellites.

Each of the $\cos p\, V_{ji}$ depends on the inclinations and can be developed according to Hill in an absolutely converging series in terms of the inclinations, since these are very small. The development (64) is thus a double series, and as it is absolutely converging the arrangement of the terms has no influence on its value, so that (64) can be regarded as a series proceeding in ascending powers of the inclinations, and absolutely converging.

In chapter II it has been shown that the only secular terms of the second degree in the inclinations in this development are:

(65) $$-\tfrac{1}{4} \frac{d_{ij}^{(1)}}{a'} (p_i^2 + q_i^2 + p_j^2 + q_j^2 - 2\, p_i p_j - 2\, q_i q_j)$$

Since the expressions (61) and (65) are the same, we have:

$$+\frac{1}{8} B_{ji}^{(1)} = \frac{1}{4}\frac{d_{ji}^{(1)}}{a'}$$

The relation (58) can be used to check the values of $d_{ji}^{(1)}$ previously found. Souillart has computed the B_{ji} with his values of the α's $\left(\alpha = \frac{a_i}{a'}\right)$. In the notation of Laplace, employed by Souillart,

$$a' B_{ji}^{(1)} = \alpha c^{(1)}$$

The values of $c^{(1)}$ are given in *Souillart* II p. 11—14. These values have been corrected, — as Souillarts α differs slightly from ours, — by means of the formula:

$$[c^{(1)}] = c^{(1)} + \frac{h}{\alpha}\alpha \frac{dc^{(1)}}{d\alpha}$$

where $\frac{dc^{(1)}}{d\alpha}$ is $c_1^{(1)}$ of Souillart, also tabulated by him. (See above, p. 52).

We found the following values:

Sat.	log $c^{(1)}$	$\frac{1}{2}[\alpha][c^{(1)}]$	$d_{ji}^{(1)}$
I—II	0·688 6949	1·534 561	1·534 556
I—III	0·210 4968	0·319 890	0·319 8896
I—IV	9·869 3788—10	0·082 9241	0·082 9241
II—III	0·684 8129	1·517 083	1·517 083
II—IV	0·140 0352	0·246 0512	0·246 0517
III—IV	0·552 2932	1·014 051	1·014 114

Remembering that $d_{ji}^{(1)}$ are computed so as to be accurate to 6 decimal places and $d_{12}^{(1)}$ to 5, the numerical agreement of $d_{ji}^{(1)}$ and $B_{ji}^{(1)}$ as required by (58) is all that can be wished, except in the case of $d_{34}^{(1)}$. In order to investigate the disagreement of $d_{34}^{(1)}$ and $\frac{1}{2} a' B_{34}^{(1)}$ we have computed directly the $c^{(1)}$ by means of table IV, add. 1re of Leverrier (*Annales de l'Obs. de Paris*, Tome II) and then $\frac{1}{2}\alpha c^{(1)}$. We found for the latter 1·014 115, so that in this case also (58) is numerically satisfied. At the same time we can conclude that $c^{(1)}$ of Souillart is erroneous. In the computation of the b's it appeared that in the combination III—IV $b^{(3)}$ and $b^{(4)}$ did not agree with Souillart's values. Considering that the b's have been computed by Souillart by means of the $c^{(1)}$ (see *Souillart* II p. 11) it is clear that the b's could not entirely agree with ours.

In the case of the Sun Souillart has not used Leverrier's development but, since in the theory of the satellites $\frac{r_i}{r_s}$ is small, he has preferred the development employed by Puiseux in his lunar theory. *) In this development everything is neglected that is of an order higher than $\left(\frac{r_i}{r_s}\right)^3$, or, which is the same,

*) Puiseux, Sur les principales inégalités du mouvement de la Lune, Annales de l'Ecole normale supérieure p. 45.

if we neglect the eccentricities, all terms higher than α_{is}^3 have been omitted. We then find according to Souillart:

$$(66) \qquad S\left[R_{si}\right] = -\frac{3}{8}\frac{a_i^2}{a_s^3}\left[p_i^2 + q_i^2 + p_s^2 + q_s^2 - 2q_iq_s - 2p_ip_s\right]$$

while according to Hill's method:

$$(67) \qquad S\left[R_{si}\right] = -\frac{1}{4}\frac{d_{si}^{(1)}}{a_s}\left[p_i^2 + q_i^2 + p_s^2 + q_s^2 - 2p_ip_s - 2q_iq_s\right]$$

If we neglect in (67) α_{si}^4 — as has also been done in the computation of $d_{si}^{(1)}$ — it must agree with (66).

Now $b^{(i)}$ is of the order α^i, and thus, omitting α_{si}^4, we have:

$$d_{si}^{(1)} = 2\,b_{si}^{(2)} = \frac{3}{2}\,\alpha_{si}^2 = \frac{3}{2}\frac{a_i}{a_s^2}$$

which shows the agreement of (67) and (66).

There is no essential difference between the methods of developing the perturbative functions as given by Leverrier and Hill. According to both methods we find a double series that can be regarded as a series in ascending powers of the inclinations. In Leverrier's method the reciprocal of the mutual distance is developed first in terms of the inclinations and then each of the coefficients in a Fourier series, while in Hill's method of procedure the order of these operations is reversed.

Hill's development is not simpler than that of Leverrier, if we only require the secular terms of the second degree in the inclinations. In this case it is even inferior to Leverrier's method, since the coefficient $B_{ji}^{(1)}$ can be found from one series, that can soon be evaluated by means of Leverrier's formulae, whereas in order to find $d_{ji}^{(1)}$ accurate to a sufficient number of decimal places, we need to compute a number of the coefficients $b^{(i)}$ of Laplace, each of which is an infinite series, that does not converge as rapidly as the series of the $B_{ji}^{(1)}$. Presumably Hill has felt this difficulty of his method and made an attempt to remove it by remarking (see *Astron. Journal* N°. 646, 647, p. 179) that experience has shown that only insignificant perturbations result from terms higher than $B^{(12)}\cos 12\,V_{ji}$ in the development (64), whence it is unnecessary to compute the $b^{(i)}$ higher than $i = 12$.

As has been pointed out on pp. 52 and 56, in order to satisfy the degree of accuracy required, we had in some cases to go as far as $b^{(32)}$.

One advantage of Hill's method over that of Leverrier is that it does away with the developments of the -3^{d}, -5th etc. powers of the distance and we only need to compute the coefficients $b^{(i)}$ of Laplace, even if we wish a complete development of the perturbative function in terms of the inclinations, for it is easily verified that the coefficients of all powers of the inclinations are functions only of the $b^{(i)}$'s, whereas in using Leverrier's method, it is necessary to compute still more coefficients of Laplace, which makes it in the long run more laborious than Hill's.

B. Comparison of the differential equations.

The equations from which we started, and which also apply to a fixed plane of (x, y), were:

$$\frac{d\psi_i}{dt} = -\frac{\partial R_i}{\partial \Psi_i}, \qquad \frac{d\Psi_i}{dt} = \frac{\partial R_i}{\partial \psi_i}$$

where $\psi_i = -\vartheta_i$ (longitude of the node), $\Psi_i = 2\,G_i \sin^2\frac{\varphi_i}{2}$, φ_i being the inclination referred to the plane of (x,y), and $G_i = n_i\,a_i^2\,\sqrt{1-e_i^2}$.

From the first equation it follows that:

$$-\frac{d\vartheta_i}{dt} = -\frac{\partial R_i}{\partial \varphi_i}\frac{d\varphi_i}{\partial \Psi_i} = -\frac{\partial R_i}{d\varphi_i}\frac{1}{n_i a_i^2\sqrt{1-e_i^2}\,\sin\varphi_i} \tag{68}$$

The second equation can also be transformed to ordinary elliptical elements, since:

$$\frac{d\Psi_i}{dt} = 2\sin^2\frac{\varphi_i}{2}\frac{dG_i}{dt} + G_i\sin\varphi_i\frac{d\varphi_i}{dt} = -\frac{\partial R_i}{\partial\vartheta_i} \tag{69}$$

Making use of the relations:

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \varepsilon}$$

$$\frac{de}{dt} = -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \varpi} - \sqrt{1-e^2}\,\frac{1-\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \varepsilon}$$

and: $n_i^2 a_i^3 = \mu_i$

it is easily verified that:

$$\frac{d\varphi_i}{dt} = -\frac{\tan\frac{\varphi_i}{2}}{n_i a_i^2\sqrt{1-e_i^2}}\left(\frac{\partial R_i}{\partial \varpi_i} + \frac{\partial R_i}{\partial \varepsilon_i}\right) - \frac{1}{n_i a_i^2\sqrt{1-e_i^2}\,\sin\varphi_i}\frac{\partial R_i}{\partial\vartheta_i} \tag{70}$$

The equations (69) and (70) are then the same equations as used by Souillart, whence the fundamental equations in both theories are the same. But the equations afterwards used do not agree, because of the different planes of (x,y) to which they have been transformed.

The equations finally used by Souillart are in a slightly modified notation:

$$\frac{dp}{d\tau} + [0]\,q - (0,1)\,q' - (0,2)\,q'' - (0,3)\,q''' - (0)\,q^{IV} = [0]\,Q$$

$$\frac{dq}{d\tau} - [0]\,p + (0,1)\,p' + (0,2)\,p'' + (0,3)\,p''' + (0)\,p^{IV} = -[0]\,P$$

etc.

$$\frac{dp^{IV}}{d\tau} - (0)\,q - (1)\,q' - (2)\,q'' - (3)\,q''' + (\,)\,q^{IV} = (5)\,Q$$

$$\frac{dq^{IV}}{d\tau} + (0)\,p + (1)\,p' + (2)\,p'' + (3)\,p''' - (\,)\,p^{IV} = -(5)\,P$$

where:

$$[0] = (0,1) + (0,2) + (0,3) + \frac{Jn}{a^2} + \frac{3}{4}\frac{n_s^2}{n}, \ \ldots\ldots \text{ etc.}$$

$$(0,1) = \frac{1}{4}\,m'\,na\,B^{(1)}, \ \ldots\ldots \text{ etc.}$$

$$(0) = \frac{3}{4}\frac{2C-A-B}{iC}\,mn^2, \ (1) = \frac{3}{4}\frac{2C-A-B}{iC}\,m'n'^2, \ \ldots\ldots \text{ etc.}^*)$$

$$(5) = \frac{3}{4}\frac{2C-A-B}{iC}\,n_s^2$$

$$(\,) = (0) + (1) + (2) + (3) + (5)$$

*) As to the meaning of i, see p. 83.

while our equations are:

$$\frac{dP_1}{d\tau} - [1]\,Q_1 - (12)\,Q_2 - (13)\,Q_3 - (14)\,Q_4 = \quad (15)\,Q_5$$

$$\frac{dQ_1}{d\tau} + [1]\,P_1 + (12)\,P_2 + (13)\,P_3 + (14)\,P_4 = -\,(15]\,P_5$$

etc.

$$\frac{dP_5}{d\tau} - \{1\}\,Q_1 - \{2\}\,Q_2 - \{3\}\,Q_3 - \{4\}\,Q_4 = \quad \{5\}\,Q_5$$

$$\frac{dQ_5}{d\tau} + \{1\}\,P_1 + \{2\}\,P_2 + \{3\}\,P_3 + \{4\}\,P_4 = -\,\{5\}\,P_5$$

where:

$$[i] = \quad \frac{2}{G_i}\,[A_i + \Sigma_j B_{ij}] + \frac{2\,C_i}{Cn}$$

$$(ij) = -\,\frac{2\,B_{ij}}{\sqrt{G_i G_j}} + \frac{2\,C_j}{Cn}\sqrt{\frac{G_i}{G_j}}$$

$$(i5) = -\,\frac{2\,B_{is}}{\sqrt{G_i}} + \frac{2\,C_s}{Cn}\sqrt{G_i}$$

$$\{i\} = \quad \frac{2\,C_i}{Cn\sqrt{G_i}}\cos i_0\ , \quad \{5\} = \frac{2\,C_s}{Cn}\cos i_0\ ,$$

$$A_i = -\,\frac{1}{2}\frac{J_i}{a_i}\mu_i\,(1 + \tfrac{3}{2}\,e_i^2) - \frac{1}{2}\frac{K_i}{a_i}\mu_i\,(1 + 5\,e_i^2)$$

$$C_i = \quad \frac{m_i}{\mu_i}A_i$$

$$\mu_i = 1 + \frac{m_i}{m_0}\ ,\ J_i = \frac{Jb^2}{a_i^2}\ ,\ K_i = \frac{Kb^4}{a_i^4}\ ,\ 4\,B_{ij} = -\,\frac{m_j}{a'}\,d_{ij}^{(1)}\ ,\ G_i = n_i a_i^2\sqrt{1-e_i^2}$$

By the relation (58) we have:

$$-\,8\,B_{ij} = m_j B_{ij}^{(1)}$$

Moreover we have:

$$d_{is}^{(1)} = \frac{3}{2}\frac{a_i^2}{a_s^2}$$

If now we introduce the same approximations as Souillart, and thus take $K_i = 0$; $e_i = 0$, and remember that between our P_i and Q_i and the p_i, q_i of Souillart there exist the relations: $P_i = q_i\sqrt{G_i}$, $Q_i = -\,p_i\sqrt{G_i}$, if $i = 1, \ldots 4$, and $P_5 = -\,q_5$, $Q_5 = p_5$ we ought to have, if both sets of differential equations referred to the same plane of (x, y):

$$-\,[1] \quad = [0] \quad , \quad -\,[2] \quad = [1] \quad , \quad \text{etc.}$$

$$\frac{\sqrt{G_2}}{\sqrt{G_1}}\,(12) \;=\; (0,1) \quad , \quad \frac{\sqrt{G_3}}{\sqrt{G_1}}\,(13) = (0,2) \quad , \quad \text{etc.} \ldots\ \frac{1}{\sqrt{G_1}}\,(15) = (0)$$

$$\sqrt{G_1}\,\{1\} \;=\; -\,(0) \quad , \quad \sqrt{G_2}\,\{2\} \;=\; -\,(1) \quad , \quad \text{etc.} \ldots\ \{5\} = -\,(\ )$$

where the symbols on the right-hand sides are those of Souillart.

Now we have:

$$-[1] = \frac{2}{n_1 a_1^2}\left[\frac{Jn_1^2}{2} + \frac{3}{4}\frac{n_s^2}{n_1} + \frac{1}{4}\Sigma_j m_j na B_{ij}^{(1)}\right] + \frac{2C_1}{Cn} \quad , \text{ if making use of the relations:}$$

$$n_i^2 a_i^3 = \mu_i \quad , \quad n_s^2 a_s^3 = m_s$$

and thus:

$$[1] = [0] + \frac{2C_1}{Cn}$$

In like manner we find:

$$\frac{\sqrt{G_2}}{\sqrt{G_1}}(12) = \frac{1}{4} m_2 na B_{12}^{(1)} + 2\frac{C_2}{Cn}$$

and since:

$$(0,1) = \frac{1}{4} m_2 na B_{12}^{(1)}$$

we find again:

$$\frac{\sqrt{G_2}}{\sqrt{G_1}}(12) = (0,1) + \frac{2C_2}{Cn} \quad , \text{ and similar identities.}$$

In the preceding the figure of Jupiter has been regarded as an ellipsoid of revolution, whence $A = B$. Consequently:

$$(0) = \frac{3}{2}\frac{C-A}{iC} mn^2 \quad , \quad (1) = \frac{3}{2}\frac{C-A}{iC} m'n'^2 \quad , \quad \ldots\ldots$$

where, as the reader may be reminded, in Souillart's notation our $n_1 = n$, $n_2 = n'$, etc., and his i is our n, the velocity of rotation of Jupiter round its axis, while:

$$(5) = \frac{3}{2}\frac{C-A}{iC} n_s^2 .$$

If we now make use of the relation:

$$Jb^2 = \frac{3}{2}(C-A)$$

our coefficients $\{i\}$ become:

$$\sqrt{G_1}\,\{1\} = \frac{-Jb^2 m_1 \cos i_0}{a_1^3 nC} = -\frac{3}{2}\frac{C-A}{nC} m_1 n_1^2 \cos i_0$$

whence:

$$\sqrt{G_i}\,\{1\} = -(0) \quad , \quad \ldots\ldots\ldots \quad , \text{ neglecting } i_0 \text{ and putting } \cos i_0 = 1 .$$

Finally we have:

$$\{5\} = \frac{2C_s}{Cn} = -\frac{Jb^2 m_s}{a_s^3 Cn} = -\frac{3}{2}\frac{C-A}{Cn} n_s^2 \quad , \text{ and thus also: } \{5\} = -(5),$$

while it is seen at a glance that the last of the corresponding coefficients $\{5\}$ and ( ) entirely disagree.

From the preceding comparison we can conclude that none of the coefficients of the equations agree exactly; it is however easily verified that this is only a consequence of the difference of the fundamental planes. With sufficient approximation, we can put:

$$p = p' + \beta$$
$$q = q' + \alpha \quad ,$$

where p' and q' are those of Souillart, and thus the coefficients of p and q in the differential equations will differ by the coefficients of $\frac{d\beta}{d\tau}$ and $\frac{d\alpha}{d\tau}$. By means of these transformation-formulae we can also explain the disagreement of $\frac{1}{\sqrt{G_1}}$ (15) and (0).

The disagreement between $\{5\}$ and ( ), can also be explained by the difference of the planes of (x, y); on referring the equations of precession and nutation to the orbit of Jupiter, each part of the perturbative function U contributes to the nutation, whence the coefficient of p^{IV} is equal to the sum of the coefficients of the other p, and similarly for q^{IV}, while in our method only the Sun's coordinates contain precession and nutation.

The difference of the corresponding coefficients in both sets of equations, although small, is not negligible as can be verified from the numerical values of $\frac{2C_i}{Cn}$, which are:

$$\frac{2C_i}{Cn} = -\ 0{\cdot}00000\ 32743 \qquad (i = 1)$$
$$= -\ 0{\cdot}00000\ 05151 \qquad (i = 2)$$
$$= -\ 0{\cdot}00000\ 04201 \qquad (i = 3)$$
$$= -\ 0{\cdot}00000\ 00426 \qquad (i = 4)$$
$$= -\ 0{\cdot}00000\ 00140 \qquad (i = s)$$

In comparison with the coefficients $[i]$ the $\frac{2C_i}{Cn}$ are negligible, while they are at the same time quite comparable with some of the coefficients (ij). The disagreement of $\{5\}$ and ( ) is large, and larger than $\{5\}$ itself; since the equation of the fifth degree furnishing the values of b_i is, with sufficient approximation, equal to the diagonal of the determinant (H), it is clear that the b_i in both methods will nearly agree, except b_5; the disagreement would be smaller if we left the motion of the equator out of account.

CHAPTER VI.

INTEGRATION OF THE DIFFERENTIAL EQUATIONS (F_i), INCLUDING TERMS OF THE THIRD DEGREE IN THE INCLINATIONS.

As it is impossible to integrate the equations (F_i) rigorously taking account of all terms of the third degree in the inclinations in the right-hand members, the only way in which the set (F_i) can be treated is by trying to integrate them by some method of approximation. In order to carry this out we can first neglect all periodic terms of the third degree in the inclinations and only take account of the secular terms; afterwards the slight corrections to be applied to the integrals because of the periodic terms are easily found.

The secular terms of the third degree in the inclinations occurring on the right-hand sides of the equations (F_i) result both from terms of the second and of the fourth degree in the inclinations of the perturbative function. The latter proceed only from the part $fm_0 \Sigma_j \frac{m_j}{\Delta_{ij}}$ of that function. If we denote the group of terms of the fourth degree of R_i bij R_i^{IV}, we find:

$$R_i^{IV} = fm_0 \Sigma_j \frac{m_j}{a'} \Big[\tfrac{1}{16} d^{(1)} \big(2 p_i^2 p_j^2 + 2 q_i^2 q_j^2 - 2 p_i^2 q_i^2 - 2 p_j^2 q_j^2 + 4 p_i p_j q_i q_j - p_i^4 - p_j^4 - q_i^4 - q_j^4 \big)$$
$$+ \tfrac{1}{32} c^{(2)} \big(p_i^4 + 2 p_i^2 q_i^2 + q_i^4 + p_j^4 + 2 p_j^2 q_j^2 + q_j^4 + 6 p_i^2 p_j^2 - 2 p_i^2 q_j^2 - 2 q_i^2 p_j^2$$
$$+ 6 q_i^2 q_j^2 + 16 p_i p_j q_i q_j - 4 p_i^3 p_j - 4 p_i p_j^3 - 4 p_i p_j q_i^2 - 4 p_i p_j q_j^2 - 4 p_i^2 q_i q_j - 4 p_j^2 q_i q_j$$
$$- 4 q_i^3 q_j - 4 q_j^3 q_i \big) \Big]$$

As could be expected this expression is symmetrical in p_i and q_i. On differentiating it with respect to p_i and q_i, we find:

$$\frac{\partial R_i^{IV}}{\partial p_i} = fm_0 \Sigma_j \frac{m_j}{a'} \Big[\tfrac{1}{16} d^{(1)} \big(4 p_i p_j^2 - 4 p_i q_i^2 + 4 p_j q_i q_j - 4 p_i^3 + \big) \tfrac{1}{32} c^{(2)} \big(4 p_i^3 + 4 p_i q_i^2 + 12 p_i p_j^2 - 4 p_i q_j^2$$
$$+ 16 p_j q_i q_j - 12 p_i^2 p_j - 4 p_j^3 - 4 p_j q_i^2 - 4 p_j q_j^2 - 8 p_i q_i q_j \big) \Big]$$

$$\frac{\partial R_i^{IV}}{\partial q_i} = fm_0 \Sigma_j \frac{m_j}{a'} \Big[\tfrac{1}{16} d^{(1)} \big(4 q_i q_j^2 - 4 p_i^2 q_i + 4 p_i p_j q_j - 4 q_i^3 \big) + \tfrac{1}{32} c^{(2)} \big(4 q_i^3 + 4 p_i^2 q_i + 12 q_i q_j^2 - 4 q_i p_j^2$$
$$+ 16 p_i p_j q_j - 12 q_i^2 q_j - 4 q_j^3 - 4 p_i^2 q_j - 4 p_j^2 q_j - 8 p_i p_j q_i \big) \Big]$$

These expressions are to be substituted in the right hand members of the equations (F_1). If we denote by R_i^{II} the complex of secular terms of the second degree in the inclinations, the equations (F_1) are:

$$\frac{dP_i}{d\tau} = \frac{1}{\sqrt{G_i}}\left[\frac{\partial R_i^{II}}{\partial q_i}\right] + \frac{1}{\sqrt{G_i}}\left[-q_i^2 - \frac{p_i^2}{2}\right]\frac{\partial R_i^{II}}{\partial q_i} - \frac{1}{2}\frac{p_i q_i}{\sqrt{G_i}}\frac{\partial R_i^{II}}{\partial p_i} + \frac{\sqrt{G_i}}{Cn}\Sigma_j \frac{\partial U_j}{\partial q_j}$$
$$- \frac{\sqrt{G_i}}{2\,Cn}\Sigma_j \left\{\alpha^2 + \beta^2 + p_i^2 + p_j^2 + q_i^2 + q_j^2\right\}\frac{\partial U_j}{\partial q_j} + \frac{1}{\sqrt{G_i}}\frac{\partial R_i^{IV}}{\partial q_i}.$$

$$\frac{dQ_i}{d\tau} = -\frac{1}{\sqrt{G_i}}\left[\frac{\partial R_i^{II}}{\partial p_i}\right] - \frac{1}{\sqrt{G_i}}\left[-p_i^2 - \frac{q_i^2}{2}\right]\frac{\partial R_i^{II}}{\partial p_i} + \frac{1}{2}\frac{p_i q_i}{\sqrt{G_i}}\frac{\partial R_i^{II}}{\partial q_i} - \frac{\sqrt{G_i}}{Cn}\Sigma_j \frac{\partial U_j}{\partial p_j}$$
$$(H_I) \qquad + \frac{\sqrt{G_i}}{2\,Cn}\Sigma_j \left\{\alpha^2 + \beta^2 + p_i^2 + q_i^2 + p_j^2 + q_j^2\right\}\frac{\partial U_j}{\partial p_j} - \frac{1}{\sqrt{G_i}}\frac{\partial R_i^{IV}}{\partial p_i}$$

where: $i = 1, \ldots 4$, $j = 1, \ldots, 5$, but in the combination ij $j \neq i$.

$$\frac{d\alpha}{d\tau} = -\frac{1}{Cn}\Sigma_i \frac{\partial U_i}{\partial p_i} + \frac{1}{2\,Cn}\Sigma_i (\alpha^2 + \beta^2 + p_i^2 + q_i^2)\frac{\partial U_i}{\partial p_i}$$
$$\frac{d\beta}{d\tau} = \frac{1}{Cn}\Sigma_i \frac{\partial U_i}{\partial q_i} - \frac{1}{2\,Cn}\Sigma_i (\alpha^2 + \beta^2 + p_i^2 + q_i^2)\frac{\partial U_i}{\partial q_i} \quad , \quad i = 1, \ldots 5$$

where:

$$R_i^{II} = (A_i + \Sigma_j B_{ij})\,p_i^2 + (A_i + \Sigma_j B_{ij})\,q_i^2 + \Sigma_j B_{ij} p_j^2 + \Sigma_j B_{ij} q_j^2 - 2\,\Sigma_j B_{ij}(p_i p_j + q_i q_j)$$
$$U_i = C_i (p_i^2 + q_i^2)$$

Substituting these expressions in the system (H_I) we finally find:

$$(H_{II}) \begin{cases} \dfrac{dP_i}{d\tau} = \dfrac{1}{\sqrt{G_i}}\left[\dfrac{\partial R_i^{II}}{\partial q_i}\right] - \dfrac{2}{\sqrt{G_i}}\left[A_i + \Sigma_j B_{ij}\right] q_i^3 + \dfrac{2}{\sqrt{G_i}}\Sigma_j B_{ij} q_i^2 q_j - \dfrac{1}{\sqrt{G_i}}\left[A_i + \Sigma_j B_{ij}\right] p_i^2 q_i \\ \quad + \dfrac{1}{\sqrt{G_i}}\left[\Sigma_j B_{ij} p_i^2 q_j\right] - \dfrac{1}{\sqrt{G_i}}\left[A_i + \Sigma_j B_{ij}\right] p_i^2 q_i + \dfrac{1}{\sqrt{G_i}}\Sigma_j B_{ij} p_i p_j q_i \\ \quad - \dfrac{\sqrt{G_i}}{Cn}\Sigma_j C_j q_j \left\{\alpha^2 + \beta^2 + p_i^2 + q_i^2 + p_j^2 + q_j^2\right\} + \dfrac{1}{\sqrt{G_i}}\dfrac{\partial R_i^{IV}}{\partial q_i}. \\ \dfrac{dQ_i}{d\tau} = -\dfrac{1}{\sqrt{G_i}}\left[\dfrac{dR_i^{II}}{\partial p_i}\right] + \dfrac{2}{\sqrt{G_i}}\left[A_i + \Sigma_j B_{ij}\right] p_i^3 - \dfrac{2}{\sqrt{G_i}}\Sigma_j B_{ij} p_i^2 p_j + \dfrac{1}{\sqrt{G_i}}\left[A_i + \Sigma_j B_{ij}\right] p_i q_i^2 \\ \quad - \dfrac{1}{\sqrt{G_i}}\left[\Sigma_j B_{ij} p_j q_i^2\right] + \dfrac{1}{\sqrt{G_i}}\left[A_i + \Sigma_j B_{ij}\right] p_i q_i^2 - \dfrac{1}{\sqrt{G_i}}\Sigma_j B_{ij} p_i q_i q_j \\ \quad + \dfrac{\sqrt{G_i}}{Cn}\Sigma_j C_j p_j \left\{\alpha^2 + \beta^2 + p_i^2 + q_i^2 + p_j^2 + q_j^2\right\} - \dfrac{1}{\sqrt{G_i}}\dfrac{\partial R_i^{IV}}{\partial p_i} \end{cases}$$

where $i = 1, \ldots 4$, $j = 1, \ldots 5$, but in the combination ij $j \neq i$

$$\frac{d\alpha}{d\tau} = -\frac{1}{Cn}\Sigma_i \frac{\partial U_i}{\partial p_i} + \frac{1}{Cn}\Sigma_i C_i (\alpha^2 + \beta^2 + p_i^2 + q_i^2)\,p_i$$
$$\frac{d\beta}{d\tau} = +\frac{1}{Cn}\Sigma_i \frac{\partial U_i}{\partial q_i} - \frac{1}{Cn}\Sigma_i C_i (\alpha^2 + \beta^2 + p_i^2 + q_i^2)\,q_i \quad , \quad i = 1, \ldots 5.$$

The right hand members contain p_5 and q_5, which are known functions of the time and of α and β. It is easily seen that the equations are less complicated if we introduce as fifth system of variables, instead of α and β, p_5 and q_5 respectively, denoted in the following by P_5 and Q_5.

We now have:

$$P_5 = p_{50}' + a'\tau + \beta$$
$$Q_5 = p_{50}' + b'\tau + \alpha$$

whence:

$$\alpha^2 = Q_5^2 - 2Q_5 x_2 + x_2^2$$
$$\beta^2 = P_5^2 - 2P_5 x_1 + x_1^2$$

where we have represented by x_1 and x_2 the quantities $p_{50}' + a'\tau$ and $q_{50}' + b'\tau$.

The method employed in order to integrate the system (H_{II}) is the method of variation of arbitrary constants. As integrals of (H_{II}) we can again take:

$$(71) \qquad \begin{aligned} P_i &= \Sigma_j \sigma_{ij} N_j \cos(b_i\tau + M_j) = \Sigma_j \sigma_{ij} K_j \\ Q_i &= \Sigma_j \sigma_{ij} N_j \sin(b_j\tau + M_j) = \Sigma_j \sigma_{ij} L_j \end{aligned}, \qquad j = 1, \ldots 5,$$

where N_j and M_j are now regarded as variables. From the equations (71) follows:

$$(72) \qquad \begin{aligned} K_j &= \Sigma_i \alpha_{ij} P_i \\ L_j &= \Sigma_i \alpha_{ij} Q_i \end{aligned}$$

where α_{ij} are functions of the constants σ_{ij} and thus are also constant.

On differentiating the equations (72) with respect to the time, we find:

$$(73) \qquad \begin{aligned} -N_j b_j \sin B_j + \frac{dN_j}{d\tau}\cos B_j - N_j \frac{dM_j}{d\tau}\sin B_j &= \Sigma_i \alpha_{ij} \frac{dP_i}{d\tau} \\ N_j b_j \cos B_i + \frac{dN_j}{d\tau}\sin B_j + N_j \frac{dM_j}{d\tau}\cos B_j &= \Sigma_i \alpha_{ij} \frac{dQ_i}{d\tau} \end{aligned}$$

where $i = 1, \ldots 5$, and $B_j = b_j\tau + M_j$

Since, if we regard N_j and M_j as constants, (71) are integrals of the equations (F_3'), which can be written in the form:

$$\frac{dP_\lambda}{d\tau} = \left[\frac{\partial R_\lambda^{II}}{\partial Q_\lambda}\right], \quad \frac{dQ_\lambda}{d\tau} = -\left[\frac{\partial R_\lambda^{II}}{\partial P_\lambda}\right],$$

where the expressions in the brackets denote the complex of the terms of the first degree in the inclinations, while the complete equations (H_{II}) are of the form: *)

$$\frac{dP_\lambda}{d\tau} = \left[\frac{\partial R_\lambda^{II}}{\partial Q_\lambda}\right] + \left[\frac{\partial R_\lambda^{IV}}{\partial Q_\lambda}\right]$$
$$\frac{dQ_\lambda}{d\tau} = -\left[\frac{\partial R_\lambda^{II}}{\partial P_\lambda}\right] - \left[\frac{\partial R_\lambda^{IV}}{\partial P_\lambda}\right]$$

We find from (73):

$$\frac{dN_j}{d\tau}\cos(b_j\tau + M_j) - N_j \frac{dM_j}{d\tau}\sin(b_j\tau + M_j) = \Sigma_i \alpha_{ij}\left[\frac{\partial R_i^{IV}}{\partial Q_i}\right]$$
$$\frac{dN_j}{d\tau}\sin(b_j\tau + M_j) + N_j \frac{dM_j}{d\tau}\cos(b_j\tau + M_j) = \Sigma_i \alpha_{ij}\left[\frac{\partial R_i^{IV}}{\partial P_i}\right]$$

and again from these:

$$(74) \qquad \begin{aligned} \frac{dN_j}{d\tau} &= \Sigma_i \alpha_{ij}\left\{\left[\frac{\partial R_i^{IV}}{\partial Q_i}\right]\cos B_j + \left[\frac{\partial R_i^{IV}}{\partial P_i}\right]\sin B_j\right\} \\ N_j \frac{dM_j}{d\tau} &= \Sigma_i \alpha_{ij}\left\{\left[\frac{\partial R_i^{IV}}{\partial Q}\right]\sin B_j - \left[\frac{\partial R_i^{IV}}{\partial P_i}\right]\cos B_j\right\} \end{aligned}$$

*) It should be noted that $\left[\frac{\partial R^{IV}}{\partial P_\lambda}\right]$ and $\left[\frac{\partial R^{IV}}{\partial Q_\lambda}\right]$ are not equal to $\frac{\partial R^{IV}}{\partial P_\lambda}$ and $\frac{\partial R^{IV}}{\partial Q_\lambda}$ respectively, but include also terms of the third order originating from R^{II}.

From these equations N_j and M_j are to be determined as functions of the time. Instead of the set of equations (H_{II}) we have thus the system (74), which for our purpose is simpler than (H_{II}), but it is still impossible to find the integrals of (74) by a direct process. We are therefore obliged to try some approximative method. According to the method, employed in the following, we substitute the expressions (71) in the right-hand members of (74). It then appears that almost all terms are of the third degree in N_j. As a first approximation we can omit periodical parts and only take account of terms that do not contain the time, so called secular-secular terms, since the greater part of the alterations of N_j and M_j are due to these terms, while the influence of the periodical parts is very slight. If necessary, the influence of the latter can be taken into account. In the numerical computations to be given below, it will appear that the influence of the secular-secular terms is already small and that therefore the periodic terms undoubtedly can be neglected. The equations (74) are then easily integrated.

On the right hand sides of (74) $\frac{\partial R_i^{IV}}{\partial p_i}$ and $\frac{\partial R_i^{IV}}{\partial q_i}$ occur. It is convenient to introduce in these expressions the following abbreviations:

$$-\frac{1}{4}\frac{m_j}{a'}\,d_{ij}^{(1)} = B_{ij}$$

$$+\frac{1}{32}\frac{m_j}{a'}\,c_{ij}^{(2)} = D_{ij}$$

We then have:

$$\frac{\partial R_i^{IV}}{\partial p_i} = -\frac{1}{4}\Sigma_j B_{ij}(4p_ip_j^2 - 4p_iq_i^2 + 4p_iq_iq_j - 4p_i^3) + \Sigma_j D_{ij}(4p_i^3 - 4p_iq_i^2 + 12p_ip_j^2 - 4p_iq_j^2 + 16p_iq_iq_j - 12p_i^2p_j - 4p_j^3 - 4p_jq_i^2 - 4p_jq_j^2 - 8p_iq_iq_j)$$

$$\frac{\partial R_i^{IV}}{\partial q_i} = -\frac{1}{4}\Sigma_j B_{ij}(4q_iq_j^2 - 4p_i^2q_i + 4p_jp_iq_j - 4q_i^3) + \Sigma_j D_{ij}(4q_i^3 - 4p_i^2q_i + 12q_iq_j^2 - 4q_ip_j^2 + 16p_ip_jq_j - 12q_i^2q_j - 4q_j^3 - 4p_i^2q_j - 4p_j^2q_j - 8p_ip_jq_i)$$

Inserting these expressions in (H_{II}) and remembering the relations, which exist between p_5, q_5, α and β, we find, if we put again:

$$P_i = p_i\sqrt{G_i}$$

$$Q_i = q_i\sqrt{G_i} \quad , \quad G_5 = 1$$

$$(H_{III})\quad \begin{cases} \dfrac{dP_i}{d\tau} = \dfrac{1}{\sqrt{G_i}}\left[\dfrac{\partial R_i^{II}}{\partial q_i}\right] + (A_i)Q_i^3 + \Sigma_\lambda(B_{i\lambda})Q_i^2Q_\lambda + (C_i)P_i^2Q_i + \Sigma_\lambda(D_{i\lambda})P_i^2Q_\lambda + \Sigma_\lambda(E_{i\lambda})P_iP_\lambda Q_i \\ \qquad + \Sigma_\lambda(F_{i\lambda})(P_\lambda^2Q_\lambda + Q_\lambda^3) + \Sigma_\lambda(H_{i\lambda})(P_5^2 + Q_5^2)Q_\lambda - 2\Sigma_\lambda(H_{i\lambda})(P_5x_1 + Q_5x_2)Q_\lambda \\ \qquad + \Sigma_\lambda(H_{i\lambda})Q_\lambda(x_1^2 + x_2^2) + \Sigma_\lambda(K_{i\lambda})P_\lambda^2Q_i + \Sigma_\lambda(L_{i\lambda})Q_\lambda^2Q_i + \Sigma_\lambda(M_{i\lambda})P_iP_\lambda Q_i \\ \qquad \text{where } \lambda = 1, 2, \ldots\ldots 5. \\ \dfrac{dQ_i}{d\tau} = -\dfrac{1}{\sqrt{G_i}}\left[\dfrac{\partial R_i^{II}}{\partial p_i}\right] - (A_i)P_i^3 - \Sigma_\lambda(B_{i\lambda})P_i^2P_\lambda - \ldots\ldots\ldots\ldots \text{ the same as } \dfrac{dP_i}{d\tau}, \text{ but with} \\ \qquad -Q_i, -P_i, -x_2 \text{ instead of } P_i, Q_i, x_1 \text{ respectively.} \end{cases}$$

Both equations hold for $i = 1, \ldots\ldots 4$. For $i = 5$ we have:

$$\begin{cases} \dfrac{dP_5}{d\tau} = +\dfrac{1}{Cn}\Sigma_\lambda\dfrac{\partial U_\lambda}{\partial q_\lambda} + b' - \Sigma_\lambda(A_{5\lambda})(P_5^2 + Q_5^2)Q_\lambda + 2\Sigma_\lambda(A_{5\lambda})(P_5x_1 + Q_5x_2)Q_\lambda \\ \qquad - \Sigma_\lambda(A_{5\lambda})Q_\lambda(x_1^2 + x_2^2) - \Sigma_\lambda(B_{5\lambda})(P_\lambda^2Q + Q_\lambda^3) \\ \dfrac{dQ_5}{d\tau} = -\dfrac{1}{Cn}\Sigma_\lambda\dfrac{\partial U_\lambda}{\partial p_\lambda} + a' + \Sigma_\lambda(A_{5\lambda})(P_5^2 + Q_5^2)P_\lambda - 2\Sigma_\lambda(A_{5\lambda})(P_5x_1 + Q_5x_2)P_\lambda \\ \qquad + \Sigma_\lambda(A_{5\lambda})P_\lambda(x_1^2 + x_2^2) + \Sigma_\lambda(B_{5\lambda})(Q_\lambda^2P_\lambda + P_\lambda^3) \\ \qquad \text{where } \lambda = 1, 2, \ldots\ldots 5. \end{cases}$$

The meaning of the different symbols is:

$$(A_i) = -\frac{1}{G_i^2}\left[2A_i + \Sigma_\lambda B_{i\lambda} - 4\Sigma_\lambda D_{i\lambda} - \frac{2C_iG_i}{Cn}\right]$$

$$(B_{i\lambda}) = -\frac{1}{G_i\sqrt{G_iG_\lambda}}\left[2B_{i\lambda} - \frac{G_i}{Cn}C_\lambda - 12D_{i\lambda}\right]$$

$$(C_i) = -\frac{1}{G_i^2}\left[2A_i + \Sigma_\lambda B_{i\lambda} + \frac{2C_iG_i}{Cn} - 4\Sigma_\lambda D_{i\lambda}\right]$$

$$(D_{i\lambda}) = \frac{1}{G_i\sqrt{G_iG_\lambda}}\left[B_{i\lambda} - \frac{G_i}{Cn}C_\lambda - 4D_{i\lambda}\right]$$

$$(E_{i\lambda}) = \frac{1}{G_i\sqrt{G_iG_\lambda}}\left[B_{i\lambda} - 8D_{i\lambda}\right]$$

$$(F_{i\lambda}) = -\frac{1}{G_\lambda^2}\left[\frac{C_\lambda\sqrt{G_iG_\lambda}}{Cn} + 4D_{i\lambda}\right]$$

$$(H_{i\lambda}) = -\frac{C_\lambda\sqrt{G_i}}{Cn\sqrt{G_\lambda}}$$

$$(K_{i\lambda}) = -\frac{4D_{i\lambda}}{G_iG_\lambda}$$

$$(L_{i\lambda}) = -\frac{1}{G_iG_\lambda}\left[B_{i\lambda} - 12D_{i\lambda}\right]$$

$$(M_{i\lambda}) = +\frac{1}{G_iG_\lambda}\left[-B_{i\lambda} + 16D_{i\lambda}\right]$$

$$(A_{5\lambda}) = \frac{1}{\sqrt{G_\lambda}}\frac{C_\lambda}{Cn}$$

$$(B_{5\lambda}) = \frac{1}{G_\lambda\sqrt{G_\lambda}}\frac{C_\lambda}{Cn}\quad,\quad \text{where } \lambda = 1, \ldots\ldots 5,\quad \lambda \neq i,\quad G_5 = 1\ .$$

The equation from which N_j can be determined is the first equation (74). If we substitute in it the terms of the third degree in the inclinations, taken from the right hand sides of (H_{III}), and at the same time for P_i and Q_i, the values:

$$P_i = \Sigma_\lambda \sigma_{i\lambda} N_\lambda \cos B_\lambda$$
$$Q_i = \Sigma_\lambda \sigma_{i\lambda} N_\lambda \sin B_\lambda$$

it appears that neither $\left[\frac{\partial R_i^{IV}}{\partial Q_i}\right]\cos B_j$, nor $\left[\frac{\partial R_i^{IV}}{\partial P_i}\right]\sin B_j$ contain secular-secular terms, whence, neglecting periodical parts, that are unimportant, we find:

(I) $$\frac{dN_j}{d\tau} = 0$$

whence N_j retains its former constant value.

In order to perform the integration of the differential equation for M_j, we thus can regard N_j as constant, which materially simplifies the integration. We have from (74):

(75) $$N_j\frac{dM_j}{d\tau} = \Sigma_i \varkappa_{ij}\left\{\left[\frac{\partial R_i^{IV}}{\partial Q_i}\right]\sin B_j - \left[\frac{\partial R_i^{IV}}{\partial P_i}\right]\cos B_j\right\}$$

This equation can be integrated in the same manner as that for N_j. Therefore we first consider

only the secular-secular parts. Introducing at the same time the above values of P_i and Q_i in (75) we find, collecting all terms from which secular parts result, the following expression:

$$
\begin{aligned}
N_j \frac{dM_j}{d\tau} = \sum_{i=1}^{i=4} \varkappa_{ij} \Big[& (A_i)\,\sigma_{ij}{}^3 N_j{}^3 \sin^4 B_j + 3\,(A_i)\sigma_{ij} N_j \sin^2 B_j \sum_{j\neq\mu} \sigma_{i\mu}{}^2 N_\mu{}^2 \sin^2 B_\mu \quad *) \\
& + \sum_\lambda (B_{i\lambda})\sigma_{\lambda j} N_j \sin^2 B_j \sum_{j\neq\mu} \sigma_{i\mu}{}^2 N_\mu{}^2 \sin^2 B_\mu + \sum_\lambda (B_{i\lambda})\sigma_{ij}{}^2 \sigma_{\lambda j} N_j{}^3 \sin^4 B_j \\
& + 2\sum_\lambda (B_{i\lambda})\sigma_{ij} N_j \sin^2 B_j \sum_{j\neq\mu} \sigma_{i\mu}\sigma_{\lambda\mu} N_\mu{}^2 \sin^2 B_\mu + \sum_\mu (C_i)\,\sigma_{i\mu}{}^2 \sigma_{ij} N_\mu{}^2 N_j \cos^2 B_\mu \sin^2 B_j \\
& + \sum_\lambda (D_{i\lambda})\sigma_{\lambda j} N_j \sin^2 B_j \sum_\mu \sigma_{i\mu}{}^2 N_\mu{}^2 \cos^2 B_\mu + \sum_\lambda (E_{i\lambda})\sigma_{ij} N_j \sin^2 B_j \sum_\mu \sigma_{i\mu}\sigma_{\lambda\mu} N_\mu{}^2 \cos^2 B_\mu \\
& + \sum_\lambda \sum_\mu (F_{i\lambda})\,\sigma_{\lambda\mu}{}^2 \sigma_{\lambda j} N_\mu{}^2 N_j \cos^2 B_\mu \sin^2 B_j + 3\sum_\lambda (F_{i\lambda})\,\sigma_{\lambda j} N_j \sin^2 B_j \sum_{j\neq\mu} \sigma_{\lambda\mu}{}^2 N_\mu{}^2 \sin^2 B_\mu \\
& + \sum_\lambda (F_{i\lambda})\,\sigma_{\lambda j}{}^3 N_j{}^3 \sin^4 B_j + \sum_\lambda (H_{i\lambda})\,\sigma_{\lambda j}\sigma_{5j}{}^2 N_j{}^3 \sin^4 B_j \\
& + \sum_\lambda (H_{i\lambda})\,\sigma_{\lambda j} N_j \sin^2 B_j \sum_{j\neq\mu} \sigma_{5\mu}{}^2 N_\mu{}^2 \sin^2 B_\mu + \sum_\lambda (H_{i\lambda})\,\sigma_{\lambda j} N_j \sin^2 B_j \sum_\mu \sigma_{5\mu}{}^2 N_\mu{}^2 \cos^2 B_\mu \\
(76) \qquad & + 2\sum_\mu (H_{i\lambda})\,\sigma_{5j} N_j \sin^2 B_j \sum_{i\neq\mu} \sigma_{5\mu}\sigma_{\lambda\mu} N_\mu{}^2 \sin^2 B_\mu + \sum_\lambda (H_{i\lambda})\,\sigma_{\lambda j} N_j (\varkappa_1{}^2 + \varkappa_2{}^2) \sin^2 B_j \\
& + \sum_\lambda (K_{i\lambda})\,\sigma_{ij} N_j \sin^2 B_j \sum_{j\neq\mu} \sigma_{5\mu}\sigma_{\lambda\mu} N_\mu{}^2 \sin^2 B_\mu + \sum_\lambda (L_{i\lambda})\,\sigma_{ij} N_j \sin^2 B_j \sum_{j\neq\mu} \sigma_{\lambda\mu}{}^2 N_\mu{}^2 \sin^2 B_\mu \\
& + \sum_\lambda (L_{i\lambda})\,\sigma_{ij}\sigma_{\lambda j}{}^2 N_j{}^3 \sin^4 B_j + 2\sum_\lambda (L_{i\lambda})\,\sigma_{\lambda j} N_j \sin^2 B_j \sum_{j\neq\mu} \sigma_{i\mu}\sigma_{\lambda\mu} N_\mu{}^2 \sin^2 B_\mu \\
& + \sum_\lambda (M_{i\lambda})\,\sigma_{\lambda j} N_j \sin^2 B_j \sum_\mu \sigma_{i\mu}\sigma_{\lambda\mu} N_\mu{}^2 \cos^2 B_\mu \Big] \\
- \varkappa_{5j} \Big[& \sum_\lambda (A_{5\lambda})\,\sigma_{\lambda j} N_j \sin^2 B_j \sum_\mu \sigma_{5\mu}{}^2 N_\mu{}^2 \cos^2 B_\mu + \sum_\lambda (A_{5\lambda})\,\sigma_{5j}{}^2 \sigma_{\lambda j} N_j{}^3 \sin^4 B_j \\
& + \sum_\lambda (A_{5\lambda})\,\sigma_{\lambda j} N_j \sin^2 B_j \sum_{j\neq\mu} \sigma_{5\mu}{}^2 N_\mu{}^2 \sin^2 B_\mu + 2\sum_\lambda (A_{5\lambda})\,\sigma_{5j} N_j \sin^2 B_j \sum_{j\neq\mu} \sigma_{5\mu}\sigma_{\lambda\mu} N_\mu{}^2 \sin^2 B_\mu \\
& + \sum_\lambda (A_{5\lambda})\,\sigma_{\lambda j} N_j (\varkappa_1{}^2 + \varkappa_2{}^2) \sin^2 B_j + \sum_\lambda \sum_\mu (B_{5\lambda})\,\sigma_{\lambda\mu}{}^2 \sigma_{\lambda j} N_\mu{}^2 N_j \cos^2 B_\mu \sin^2 B_j \\
& + \sum_\lambda (B_{5\lambda})\,\sigma_{\lambda j}{}^3 N_j{}^3 \sin^4 B_j + 3\sum_\lambda (B_{5\lambda})\,\sigma_{\lambda j} N_j \sin^2 B_j \sum_{j\neq\mu} \sigma_{\lambda\mu}{}^2 N_\mu{}^2 \sin^2 B_\mu \Big] \\
& + \text{the same terms, replacing sin. by cos. and cos. by sin,} \\
& \qquad \text{and throughout } \lambda = 1, \ldots 5 \quad , \quad \mu = 1, \ldots 5 .
\end{aligned}
$$

By means of the trigonometric relations:

$$\cos^2\alpha = \tfrac{1}{2}(1 + \cos 2\alpha)$$

$$\sin^2\alpha = \tfrac{1}{2}(1 - \cos 2\alpha)$$

$$\cos^4\alpha = \tfrac{1}{8}(3 + 4\cos 2\alpha + \cos 4\alpha)$$

$$\sin^4\alpha = \tfrac{1}{8}(3 - 4\cos 2\alpha + \cos 4\alpha)$$

we find for the constant part of the right hand member of (76):

$$
\begin{aligned}
N_j \frac{dM_j}{d\tau} = \sum_{i=1}^{i=4} \alpha_{ij} \Big[& \tfrac{3}{4}(A_i)\sigma_{ij}N_j^3 + \tfrac{3}{2}(A_i)\sigma_{ij}N_j \sum_{j\neq\mu}\sigma_{i\mu}^2 N_\mu^2 + \tfrac{1}{2}\sum_\lambda (B_{i\lambda})\sigma_{\lambda j}N_j \sum_{j\neq\mu}\sigma_{i\mu}^2 N_\mu^2 + \tfrac{3}{4}\sum_\lambda (B_{i\lambda})\sigma_{ij}^2\sigma_{\lambda j}N_j^3 \\
& + \sum_\lambda (B_{i\lambda})\sigma_{ij}N_j \sum_{j\neq\mu}\sigma_{i\mu}\sigma_{\lambda\mu}N_\mu^2 + \tfrac{1}{2}\sum_\mu (C_i)\sigma_{i\mu}^2\sigma_{ij}N_\mu^2 N_j + \tfrac{1}{2}\sum_\lambda (D_{i\lambda})\sigma_{\lambda j}N_j \sum_\mu \sigma_{i\mu}^2 N_\mu^2 \\
& + \tfrac{1}{2}\sum_\lambda (E_{i\lambda})\sigma_{ij}N_j \sum_\mu \sigma_{i\mu}\sigma_{\lambda\mu}N_\mu^2 + \tfrac{1}{2}\sum_\lambda\sum_\mu (F_{i\lambda})\sigma_{\lambda\mu}^2\sigma_{\lambda j}N_\mu^2 N_j + \tfrac{3}{2}\sum_\lambda (F_{i\lambda})\sigma_{\lambda j}N_j \sum_{j\neq\mu}\sigma_{\lambda\mu}^2 N_\mu^2 \\
& + \tfrac{3}{4}\sum_\lambda (F_{i\lambda})\sigma_{\lambda j}^3 N_j^3 + \sum_\lambda (H_{i\lambda})\sigma_{\lambda j}N_j \sum_{j\neq\mu}\sigma_{5\mu}^2 N_\mu^2 + \sum_\lambda (H_{i\lambda})\sigma_{5j}N_j \sum_{i\neq\mu}\sigma_{5\mu}\sigma_{\lambda\mu}N_\mu^2 \\
& + \tfrac{5}{4}\sum_\lambda (H_{i\lambda})\sigma_{\lambda j}\sigma_{5j}^2 N_j^3 + \sum_\lambda (H_{i\lambda})\sigma_{\lambda j}N_j(\varkappa_1^2 + \varkappa_2^2) + \tfrac{1}{2}\sum_\lambda (K_{i\lambda})\sigma_{ij}N_j \sum_\lambda \sigma_{\lambda\mu}^2 N_\mu^2 \\
(77) \qquad & + \tfrac{1}{2}\sum_\lambda (L_{i\lambda})\sigma_{ij}N_j \sum_{j\neq\mu}\sigma_{\lambda\mu}^2 N_\mu^2 + \tfrac{3}{4}\sum_\lambda (L_{i\lambda})\sigma_{ij}\sigma_{\lambda j}^2 N_j^3 + \sum_\lambda (L_{i\lambda})\sigma_{\lambda j}N_j \sum_{j\neq\mu}\sigma_{i\mu}\sigma_{\lambda\mu}N_\mu^2 \\
& + \tfrac{1}{2}\sum_\lambda (M_{i\lambda})\sigma_{\lambda j}N_j \sum_\mu \sigma_{i\mu}\sigma_{\lambda\mu}N_\mu^2 \Big] - \\
- \alpha_{5j} \Big[& \sum_\lambda (A_{5\lambda})\sigma_{\lambda j}N_j \sum_{j\neq\mu}\sigma_{5\mu}^2 N_\mu^2 + \tfrac{5}{4}\sum_\lambda (A_{5\lambda})\sigma_{5j}^2\sigma_{\lambda j}N_j^3 + \sum_\lambda (A_{5\lambda})\sigma_{5j}N_j \sum_{j\neq\mu}\sigma_{5\mu}\sigma_{\lambda\mu}N_\mu^2 \\
& + \sum_\lambda (A_{5\lambda})\sigma_{\lambda j}N_j(\varkappa_1^2 + \varkappa_2^2) + \tfrac{1}{2}\sum_\lambda\sum_\mu (B_{5\lambda})\sigma_{\lambda\mu}^2\sigma_{\lambda j}N_\mu^2 N_j + \tfrac{3}{4}\sum_\lambda (B_{5\lambda})\sigma_{\lambda j}^3 N_j^3 \\
& + \tfrac{3}{2}\sum_\lambda (B_{5\lambda})\sigma_{\lambda j}N_j \sum_{j\neq\mu}\sigma_{\lambda\mu}^2 N_\mu^2 \Big]
\end{aligned}
$$

The value of $N_j \frac{dM_j}{d\tau}$ is very complicated, but it is possible to simplify this expression, since there are among the σ_{ij} very small ones and also because of the small values of the proper inclinations N_j. If we denote the right-hand side of (77) by the constant K_j, (77) assumes the form:

$$N_j \frac{dM_j}{d\tau} = K_j$$

or, since, as was pointed out above, N_j is constant:

$$\frac{dM_j}{d\tau} = \frac{K_j}{N_j}$$

and on integrating we find from this equation the correction ΔM_j of M_j, viz:

$$\Delta M_j = \frac{K_j}{N_j}\tau + \varepsilon_j = \delta_j\tau + \varepsilon_j$$

and thus we have:

(II) $$B_j = (b_j + \delta_j)\,\tau + \varepsilon_j + M_j$$

It appears from (I) and (II) that, if in the differential equations we regard not only the terms of the first but also those of the third degree in the inclinations, the integrals of the reduced system of equations retain their original form; only the coefficient of τ in the argument $b_j\tau$ undergoes a correction that depends on the masses and other elements and also on the proper inclinations.

This correction δ_j can also be found in a somewhat different way. The values of δ found by the second method can then serve as a check on the former and are therefore derived in the following.

Second method of deriving the integrals of the differential equations (H_{III}).

The integrals of the system of equations (F_3'), neglecting the right hand sides, are:

$$P_i = \sum_\lambda \sigma_{i\lambda} K_\lambda$$
$$Q_i = \sum_\lambda \sigma_{i\lambda} L_\lambda \tag{78}$$

where: $$K_\lambda = N_\lambda \cos(b_\lambda \tau + M_\lambda) \quad , \quad L_\lambda = N_\lambda \sin(b_\lambda \tau + M_\lambda).$$

In the third chapter it has been shown that these expressions are also the integrals of the complete equations (F_3'), provided that the constants N_λ and M_λ undergo slight periodical corrections.

It has just now been found that the influence of the terms of the third degree in the inclinations amounts to a correction of b_j. In order to find this, we proceed as follows:

From equations (71) we find:

$$K_j = \Sigma_i \alpha_{ij} P_i$$
$$L_j = \Sigma_i \alpha_{ij} Q_i \tag{79}$$

where α_{ij} are constants.

Instead of regarding P_i and Q_i as the variables to be determined, we can take K_j and L_j. Differentiating (79) with respect to the time, we find:

$$\frac{dK_j}{d\tau} = \Sigma_i \alpha_{ij} \frac{dP_i}{d\tau}$$
$$\frac{dL_j}{d\tau} = \Sigma_i \alpha_{ij} \frac{dQ_i}{d\tau} \tag{80}$$

The equations (80) replace the set (F_3'). The integrals (78) of (F_3') are also integrals of (80); since substituting them in (80) we find:

$$\frac{dK_j}{d\tau} = \Sigma_i \alpha_{ij} \left(\Sigma_\lambda \sigma_{i\lambda} \frac{dK_\lambda}{d\tau} \right)$$
$$\frac{dL_j}{d\tau} = \Sigma_i \alpha_{ij} \left(\Sigma_\lambda \sigma_{i\lambda} \frac{dL_\lambda}{d\tau} \right)$$

whence on integration:

$$K_j = \Sigma_i \alpha_{ij} (\Sigma_\lambda \sigma_{i\lambda} K_\lambda)$$
$$L_j = \Sigma_i \alpha_{ij} (\Sigma_\lambda \sigma_{i\lambda} L_\lambda)$$

which are identities.

On substituting (78) in (80) we must find:

$$\frac{dK_j}{d\tau} = - b_j L_j$$
$$\frac{dL_j}{d\tau} = + b_j K_j$$

If we now replace the equations (F_3') by (H_{III}) and thus take account of the terms of the third degree in the inclinations, we again find instead of (H_{III}) the same equations (80), where the differential

coefficients on the right hand sides now result from (H_{III}). The integrals (78) are still integrals of (H_{III}) and consequently of (80) provided that N_λ and M_λ are variables. If we substitute them in (80) the left hand sides of (80) become:

$$(81) \qquad -b_j N_j \sin(b_j\tau + M_j) - N_j \frac{dM_j}{d\tau} \sin(b_j\tau + M_j) + \frac{dN_j}{d\tau} \cos(b_j\tau + M_j)$$

The right hand members are then very intricate but they must be equal to (81). The terms of the first degree in the inclinations will give a result that is equal to $-b_j N_j \sin(b_j\tau + M_i)$, while the terms containing $\sin(b_j\tau + M_j)$ and resulting from parts of the third degree in the inclinations, will give the coefficient $-N_j \frac{dM_j}{d\tau}$, and those containing $\cos(b_j\tau + M_j)$ the value of $\frac{dN_j}{d\tau}$. In this manner we can test, whether $\frac{dN_j}{d\tau}$ contains only periodic parts as is shown by the preceding method, while for $N_j \frac{dM_j}{d\tau}$ we must find a constant and a periodic part. If we neglect as a first approximation the periodic parts on the right hand sides of (80) and take only terms contraining $\sin(b_j\tau + M_j)$ with a constant coefficient, the result of the substitution will be equal to:

$$(82) \qquad -\Sigma_A \Pi(N) \sin(b_j\tau + M_j) - b_j N_j \sin(b_j\tau + M_j)$$

where $\Sigma_A \Pi(N)$ depends on the proper inclinations N_j and on the constants α_{ij} and $\sigma_{i\lambda}$.

From (81) and (82) we find:

$$N_j \frac{dM_j}{d\tau} = \Sigma_A \Pi(N)$$

whence, since N_j is constant like the right-hand member, we have by integrating:

$$\delta M_j = \frac{\Sigma_A \Pi(N)}{N_j} \tau$$

The correction Δb_j of b_j thus is:

$$\frac{\Sigma_A \Pi(N)}{N_j}.$$

In order to evaluate $\Sigma_A \Pi(N)$ we can use either of the two equations (80). Using the first one, we have:

$$\begin{aligned}\frac{dK_j}{d\tau} = \sum_{i=1}^{i=4} \alpha_{ij} \Big[&\frac{1}{\sqrt{Gi}} \Big[\frac{\partial R_i''}{\partial q_i}\Big] + (A_i) Q_i^3 + \sum_\lambda (B_{i\lambda}) Q_i^2 Q_\lambda + (C_i) P_i^2 Q_i + \sum_\lambda (D_{i\lambda}) P_i^2 Q_\lambda + \sum_\lambda (E_{i\lambda}) P_i P_\lambda Q_i \\ &+ \sum_\lambda (F_{i\lambda})(P_\lambda^2 Q_\lambda + Q_\lambda^3) + \sum_\lambda (H_{i\lambda})(P_5^2 + Q_5^2) Q_\lambda - 2 \sum_\lambda (H_{i\lambda})(P_5 \varkappa_1 + Q_5 \varkappa_2) Q_\lambda \\ &+ \sum_\lambda (H_{i\lambda}) Q_\lambda (\varkappa_1^2 + \varkappa_2^2) + \sum_\lambda (K_{i\lambda}) P_\lambda^2 Q_i + \sum_\lambda (L_{i\lambda}) Q_\lambda^2 Q_i + \sum_\lambda (M_{i\lambda}) P_i P_\lambda Q_\lambda \Big] \\ + \alpha_{5j} \Big[&+ \frac{1}{Cn} \sum_\lambda \frac{\partial U_\lambda}{\partial q_\lambda} + b' - \sum_\lambda (A_{5\lambda})(P_5^2 + Q_5^2) Q_\lambda + 2 \sum_\lambda (A_{5\lambda})(P_5 \varkappa_1 + Q_5 \varkappa_2) Q_\lambda \\ &- \sum_\lambda (A_{5\lambda}) Q_\lambda (\varkappa_1^2 + \varkappa_2^2) - \sum_\lambda (B_{5\lambda})(P_\lambda^2 Q_\lambda + Q_\lambda^3) \Big]\end{aligned}$$

where $\lambda = 1, \cdots\cdots 5$.

On substituting in this equation the values (78) of P_i and Q_i and retaining only terms that contribute to $\Sigma A \Pi(N) \sin(b_j + M_j)$ we find for this group of terms:

$$
\begin{aligned}
&\sum_{i=1}^{i=4} \alpha_{ij} \Big[(A_i)\sigma_{ij}^3 N_j^3 \sin^3 B_j + 3(A_i)\sigma_{ij} N_j \sin B_j \sum_{j\neq\mu} \sigma_{i\mu}^2 N_\mu^2 \sin^2 B_\mu \\
&\quad + \sum_\lambda (B_{i\lambda})\sigma_{\lambda j} N_j \sin B_j \sum_{j\neq\mu} \sigma_{i\mu}^2 N_\mu^2 \sin^2 B_\mu + \sum_\lambda (B_{i\lambda})\sigma_{\lambda j}\sigma_{ij}^2 N_j^3 \sin^3 B_j \\
&\quad + \sum_\lambda (B_{i\lambda})\sigma_{ij} N_j \sin B_j \sum_{j\neq\mu} \sigma_{i\mu}\sigma_{\lambda\mu} N_\mu^2 \sin^2 B_\mu + \tfrac{1}{2}\sum_\mu (C_i)\sigma_{ij} N_j \sin B_j \sigma_{i\mu}^2 N_\mu^2 \cos^2 B_\mu \\
&\quad + \sum_\lambda (D_{i\lambda})\sigma_{\lambda j} N_j \sin B_j \sum_\mu \sigma_{i\mu}^2 N_\mu^2 \cos^2 B_\mu + \sum_\lambda (E_{i\lambda})\sigma_{ij} N_j \sin B_j \sum_\mu \sigma_{i\mu}\sigma_{\lambda\mu} N_\mu^2 \cos^2 B_\mu \\
&\quad + \sum_\lambda (F_{i\lambda})\sigma_{\lambda j}^3 N_j^3 \sin^3 B_j + \sum_\lambda (F_{i\lambda})\sigma_{\lambda j} N_j \sin B_j \sum_\mu \sigma_{\lambda\mu}^2 N_\mu^2 \cos^2 B_\mu \\
&\quad + 3\sum_\lambda (F_{i\lambda})\sigma_{\lambda j} N_j \sin B_j \sum_{j\neq\mu} \sigma_{\lambda\mu}^2 N_\mu^2 \sin^2 B_\mu + \sum_\lambda (H_{i\lambda})\sigma_{\lambda j}\sigma_{sj}^2 N_j^3 \sin^3 B_j \\
&\quad + \sum_\lambda (H_{i\lambda})\sigma_{\lambda j} N_j \sin B_j \sum_\mu \sigma_{s\mu}^2 N_\mu^2 \cos^2 B_\mu + \sum_\lambda (H_{i\lambda})\sigma_{\lambda j} N_j \sin B_j \sum_{j\neq\mu} \sigma_{s\mu}^2 N_\mu^2 \sin^2 B_\mu \\
&\quad + 2\sum_\lambda (H_{i\lambda})\sigma_{sj} N_j \sin B_j \sum_{j\neq\mu} \sigma_{s\mu}\sigma_{\lambda\mu} N_\mu^2 \sin^2 B_\mu + \sum_\lambda (H_{i\lambda})\sigma_{\lambda j} N_j \sin B_j (\varkappa_1^2 + \varkappa_2^2) \\
&\quad + \sum_\lambda (K_{i\lambda})\sigma_{ij} N_j \sin B_j \sum_\mu \sigma_{i\mu}^2 N_\mu^2 \cos^2 B_\mu + \sum_\lambda (L_{i\lambda})\sigma_{ij} N_j \sin B_j \sum_{j\neq\mu} \sigma_{\lambda\mu}^2 N_\mu^2 \sin^2 B_\mu \\
&\quad + \sum_\lambda (L_{i\lambda})\sigma_{ij}\sigma_{\lambda j}^2 N_j^3 \sin^3 B_j + 2\sum_\lambda (L_{i\lambda})\sigma_{\lambda j} N_j \sin B_j \sum_{j\neq\mu} \sigma_{i\mu}\sigma_{\lambda\mu} N_\mu^2 \sin^2 B_\mu \\
&\quad + \sum_\lambda (M_{i\lambda})\sigma_{\lambda j} N_j \sin B_j \sum_\mu \sigma_{i\mu}\sigma_{\lambda\mu} N_\mu^2 \cos^2 B_\mu \Big] \qquad (83) \\
&- \alpha_{sj} \Big[\sum_\lambda (A_{s\lambda})\sigma_{\lambda j} N_j \sin B_j \sum_\mu \sigma_{s\mu}^2 N_\mu^2 \cos^2 B_\mu + \sum_\lambda (A_{s\lambda})\sigma_{\lambda j} N_j \sin B_j \sum_{j\neq\mu} \sigma_{s\mu}^2 N_\mu^2 \sin^2 B_\mu \\
&\quad + \sum_\lambda (A_{s\lambda})\sigma_{\lambda j}\sigma_{sj}^2 N_j^3 \sin^3 B_j + 2\sum_\lambda (A_{s\lambda})\sigma_{sj} N_j \sin B_j \sum_{j\neq\mu} \sigma_{s\mu}\sigma_{\lambda\mu} N_\mu^2 \sin^2 B_\mu \\
&\quad + \sum_\lambda (A_{s\lambda})\sigma_{\lambda j} N_j (\varkappa_1^2 + \varkappa_1^2) \sin B_j + \sum_\lambda (B_{s\lambda})\sigma_{\lambda j}^3 N_j^3 \sin^3 B_j \\
&\quad + \sum_\lambda (B_{s\lambda})\sigma_{\lambda j} N_j \sin B_j \sum_\mu \sigma_{\lambda\mu}^2 N_\mu^2 \cos^2 B_\mu + 3\sum_\lambda (B_{s\lambda})\sigma_{\lambda j} N_j \sin B_j \sum_{j\neq\mu} \sigma_{\lambda\mu}^2 N_\mu^2 \sin^2 B_\mu \Big]
\end{aligned}
$$

Using the trigonometric relations:

$$\sin^2\alpha = \tfrac{1}{2}(1 - \cos 2\alpha)$$

$$\cos^2\alpha = \tfrac{1}{2}(1 + \cos 2\alpha)$$

$$\sin^3\alpha = \tfrac{1}{4}(3\sin\alpha - \sin 3\alpha)$$

it appears, that the constant part of (83), denoted in the preceding by $\Sigma A \Pi(N)$, exactly agrees with (77).

Approximate expression for the correction Δb_j.

Although the σ_{ij} are small except in the case of $i = j$ — the largest value being 0·2028 — their amounts are very different. Consequently, if we wish to take account of all the terms that are of the same magnitude, even an approximate expression of Δb_j would remain very complicated. In order to

get a simpler expression of Δb_j, which makes it possible to make a numerical estimate, we neglect all terms that are smaller than $\sigma_{ij} N_j^3$, if $j \neq 5$, since N_5 is much larger than the other N_j. In the case of $j = 5$, we still have to consider a few more terms containing higher powers of σ_{ij}. By proceeding in this way the most important terms are taken into account, while the neglected terms are of the same amount as the smallest terms that have been retained. We then find:

$$\begin{aligned}
\Delta b_j = \frac{1}{N_j}\Bigg[& \tfrac{3}{4}\alpha_{jj}(A_j)N_j^3 + \sum_{i=1}^{i=4}\alpha_{ij}\Big[\Big\{\tfrac{3}{2}(A_i)'\sigma_{ij} + \tfrac{1}{2}\sum_\lambda (B_{i\lambda})'\sigma_{\lambda j} + \tfrac{1}{2}(C_i)\sigma_{ij} + \tfrac{1}{2}\sum_\lambda (D_{i\lambda})\sigma_{ij}\Big\} N_i^2 N_j \\
& + \Big\{\tfrac{1}{2}\sum_\lambda (F_{i\lambda})\sigma_{\lambda j} + \tfrac{3}{2}\sum_\lambda (F_{i\lambda})'\sigma_{\lambda j} + \tfrac{1}{2}\sum_\lambda (K_{i\lambda})\sigma_{ij} + \tfrac{1}{2}\sum_\lambda (L_{i\lambda})'\sigma_{ij}\Big\} N_j N_\lambda^2 + \tfrac{3}{4}(F_{ij})N_j^3 \\
& + \tfrac{3}{4}(L_{ij})\sigma_{ij}N_j^3 + \Big\{\sum_\lambda (H_{i\lambda})'\sigma_{\lambda j} + (H_{i5})'\sigma_{5j}\Big\} N_j N_5^2 + \sum_\lambda (H_{i\lambda})\sigma_{\lambda j} N_j (x_1^2 + x_2^2) + \tfrac{1}{2}(M_{ij})\sigma_{ij}N_j^3\Big] \\
& + \tfrac{3}{4}\alpha_{jj}\sum_\lambda (B_{j\lambda})\sigma_{\lambda j}N_j^3 - \alpha_{5j}\Big[\Big\{\sum_\lambda (A_{5\lambda})\sigma_{\lambda j} + (A_{55})\sigma_{5j}\Big\} N_j N_5^2 + \sum_\lambda (A_{5\lambda})\sigma_{\lambda j} N_j (x_1^2 + x_2^2) \\
& + \Big\{\tfrac{1}{2}\sum_\lambda (B_{5\lambda})\sigma_{\lambda j} + \tfrac{3}{2}\sum_\lambda (B_{5\lambda})'\sigma_{\lambda j}\Big\} N_j N_\lambda^2 + \tfrac{3}{4}(B_{5j})N_j^3\Big]\Bigg] \qquad \text{if } j = 1, \ldots 4
\end{aligned}$$

where the accents denote that the indices of the N's must not take the same value.

We further have:

$$\begin{aligned}
\Delta b_5 = \frac{1}{N_5}\Bigg[& \sum_{i=i}^{i=4}\alpha_{i5}\Big[\Big\{\tfrac{3}{4}(A_i)\sigma_{i5}^3 N_5^3 + \tfrac{3}{2}(A_i)\sigma_{ij} + \tfrac{1}{2}\sum_\lambda (B_{i\lambda})\sigma_{\lambda 5} + \tfrac{1}{2}(C_i)\sigma_{i5} + \tfrac{1}{2}\sum_\lambda (D_{i\lambda})\sigma_{i5}\Big\} N_i^2 N_5 \\
& + \tfrac{3}{4}\sum_\lambda (B_{i\lambda})\sigma_{i5}^2\sigma_{\lambda 5}N_5^3 + \tfrac{1}{2}(C_i)\sigma_{i5}^3 N_5^3 + \tfrac{1}{2}\sum_\lambda (D_{i\lambda})\sigma_{\lambda 5}\sigma_{i5}^2 N_5^3 + \tfrac{1}{2}\sum_\lambda (E_{i\lambda})\sigma_{i5}^2\sigma_{\lambda 5}N_5^3 + \Big(\tfrac{1}{2}\sum_\lambda (F_{i\lambda})\sigma_{\lambda 5} \\
& + \tfrac{3}{2}\sum_\lambda (F_{i\lambda})'\sigma_{\lambda 5} + \tfrac{1}{2}\sum_\lambda (K_{i\lambda})'\sigma_{i5} + \tfrac{1}{2}\sum_\lambda (L_{i\lambda})'\sigma_{i5}\Big) N_5 N_\lambda^2 + \tfrac{5}{4}\sum_\lambda (H_{i\lambda})\sigma_{\lambda 5}N_5^3 + \tfrac{1}{2}\sum_\lambda (K_{i\lambda})\sigma_{i5}\sigma_{\lambda 5}^2 N_5^3 \\
& + \tfrac{1}{2}\sum_\lambda (F_{i\lambda})\sigma_{\lambda 5}^3 N_5^3 + \tfrac{3}{4}\sum_\lambda (F_{i\lambda})\sigma_{\lambda 5}^3 N_5^3 + \tfrac{5}{4}\sum_\lambda (H_{i\lambda})\sigma_{\lambda 5}N_5^3 + \sum_\lambda (H_{i\lambda})\sigma_{\lambda j}N_j(x'^2_1 + x'^2_2) \\
& + \tfrac{3}{4}\sum_\lambda (L_{i\lambda})\sigma_{i5}\sigma_{\lambda 5}^2 N_5^3 + \tfrac{1}{2}\sum_\lambda (M_{i\lambda})\sigma_{\lambda 5}^2\sigma_{i5}N_5^3\Big] \\
& - \alpha_{j5}\Big[\sum_\lambda (A_{5\lambda})\sigma_{\lambda 5}N_5^3 + \tfrac{5}{4}\sum_\lambda (A_{5\lambda})\sigma_{\lambda 5}N_5^3 + \sum_\lambda (A_{5\lambda})\sigma_{\lambda 5}N_5(x_1^2 + x_2^2) + \tfrac{1}{2}\sum_\lambda (B_{5\lambda})\sigma_{\lambda 5}^3 N_5^3 \\
& + \tfrac{3}{4}\sum_\lambda (B_{5\lambda})\sigma_{\lambda 5}^3 N_5^3\Big]\Bigg]
\end{aligned}$$

The values of the different constants are given in the following tables:

i	N_i
1	+ 0·000 3025
2	+ 0·007 4420
3	+ 0·003 2834
4	+ 0·005 1654
5	+ 0·016 685

j	D_{1j}	j	D_{2j}	j	D_{3j}	j	D_{4j}
2	+ 0·00000 84521	1	+ 0·00001 33185	1	+ 0·00000 03605	1	+ 0·00000 001376
3	+ 0·00000 07581	3	+ 0·00001 71573	2	+ 0·00000 51786	2	+ 0·00000 007694
4	+ 0·00000 00159	4	+ 0·00000 014067	4	+ 0·00000 23996	3	+ 0·00000 43484
5	negligible	5	negligible	5	+ 0·00000 0000001	5	+ 0·00000 0000010

i	(A_i)
1	+ 0·0041 2533
2	+ 0·0009 02510
3	+ 0·0001 46695
4	+ 0·0000 29067

j	(B_{1j})	j	(B_{2j})	j	(B_{3j})	j	(B_{4j})
2	— 0·00029 3545	1	— 0·00036 4927	1	— 0·00001 10688	1	+ 0·00000 05052
3	— 0·00004 36875	3	— 0·00037 5656	2	— 0·00008 95435	2	— 0·00000 16718
4	— 0·00000 21334	4	— 0·00000 58643	4	— 0·00003 58455	3	— 0·00004 88278
5	— 0·00000 06181	5	— 0·00000 11125	5	— 0·00000 20017	5	— 0·00000 40635

i	(C_i)
1	+ 0·0039 5035
2	+ 0·0005 16986
3	+ 0·0000 84233
4	+ 0·0000 09137

j	(D_{1j})	j	(D_{2j})	j	(D_{3j})	j	(D_{4j})
2	— 0·0001 09511	1	— 0·0001 38688	1	— 0·00000 77172	1	— 0·00000 22028
3	— 0·0000 192219	3	— 0·0001 39958	2	— 0·00003 35683	2	— 0·00000 10989
4	— 0·0000 010464	4	— 0·0000 02620	4	— 0·00001 38242	3	— 0·00001 90476
5	— 0·0000 003222	5	— 0·0000 00568	4	— 0·00000 10113	5	— 0·00000 20405

j	(E_{1j})	j	(E_{2j})	j	(E_{3j})	j	(E_{4j})
2	— 0·0001 84761	1	— 0·0002 30859	1	— 0·00000 74633	1	— 0·00000 046251
3	— 0·0000 24993	3	— 0·0002 36167	2	— 0·00005 64401	2	— 0·00000 107305
4	— 0·0000 00113	4	— 0·0000 03286	4	— 0·00002 20581	3	— 0·00003 01433
5	— 0·0000 00313	5	— 0·0000 00560	5	— 0·00000 10044	5	— 0·00000 203447

j	(F_{1j})	j	(F_{2j})	j	(F_{3j})	j	(F_{4j})
2	− 0·00005 31053	1	− 0·0001 30887	1	− 0·00000 034782	1	+ 0·00000 363236
3	− 0·00000 28367	3	− 0·0000 67763	2	− 0·00003 21924	2	− 0·00000 006717
4	− 0·00000 00250	4	− 0·0000 003044	4	− 0·00000 539010	3	− 0·00001 69803
5	+ 0·00000 000556	5	+ 0·0000 000062	5	+ 0·00000 000698	5	+ 0·00000 000697

j	(H_{1j})	j	(H_{2j})	j	(H_{3j})	j	(H_{4j})
2	+ 0·00000 022934	1	+ 0·00000 18385	1	+ 0·00000 206614	1	+ 0·00000 23793
3	+ 0·00000 016645	3	+ 0·00000 01869	2	+ 0·00000 028943	2	+ 0·00000 03743
4	+ 0·00000 001466	4	+ 0·00000 001646	4	+ 0·00000 001850	3	+ 0·00000 02419
5	+ 0·00000 000556	5	+ 0·00000 000624	5	+ 0·00000 000702	5	+ 0·00000 00081

j	(K_{1j})	j	(K_{2j})	j	(K_{3j})	j	(K_{4j})
2	− 0·00006 73332	1	− 0·00010 6101	1	− 0·00000 227426	1	− 0·00000 006547
3	− 0·00000 47818	3	− 0·00008 58180	2	− 0·00002 59024	2	− 0·00000 029020
4	− 0·00000 07597	4	− 0·00000 05306	4	− 0·00000 71662	3	− 0·00001 29863
5	0	5	0	5	− 0·00000 00000042	5	− 0·00000 0000030

j	(L_{1j})	j	(L_{2j})	j	(L_{3j})	j	(L_{4j})
4	+ 0·00023 1862	1	+ 0·00036 5351	1	+ 0·00000 11693	1	+ 0·00000 07377
3	+ 0·00002 45857	3	+ 0·00029 5964	2	+ 0·00008 93308	2	+ 0·00000 16789
4	+ 0·00000 08561	4	+ 0·00000 30695	4	+ 0·00002 40052	3	+ 0·00004 76982
5	+ 0·00000 024897	6	+ 0·00000 04998	3	+ 0·00000 10069	5	+ 0·00000 23487

j	(M_{1j})	j	(M_{2j})	j	(M_{3j})	j	(M_{4j})
2	+ 0·00029 9196	1	+ 0·00047 1451	1	+ 0·00001 39675	1	+ 0·00000 08031
3	+ 0·00002 93675	3	+ 0·00038 1782	2	+ 0·00011 5108	2	+ 0·00000 19691
4	+ 0·00000 09319	4	+ 0·00000 36001	4	+ 0·00003 34874	3	+ 0·00006 06844
5	+ 0·00000 02489	5	+ 0·00000 04998	5	+ 0·00000 10069	5	+ 0·00000 23488

j	(A_{5j})	j	(B_{5j})
1	− 0·00000 20610	1	− 0·00000 32663
2	− 0·00000 02887	2	− 0·00000 04568
3	− 0·00000 02095	3	− 0·00000 02085
4	− 0·00000 00184	4	− 0·00000 00138
5	− 0·00000 00140	5	− 0·00000 00140

Numerical values of the constants α_{ij} and of the corrections Δb_i.

Remembering that α_{ij} are the constant coefficients in:

$$K_j = \Sigma\, \alpha_{ij}\, P_i$$
$$L_j = \Sigma\, \alpha_{ij}\, Q_i$$

derived from:

$$P_i = \Sigma\, \sigma_{ij}\, K_j$$
$$Q_i = \Sigma\, \sigma_{ij}\, L_j \quad ,$$

we find by means of the values of the σ_{ij}:

i	α_{i1}	α_{i2}	α_{i3}	α_{i4}	α_{i5}
1	+ 0·991 215	— 0·025 594	— 0·000 496	— 0·001 384	— 0·001 332
2	— 0·018 194	+ 1·014 470	+ 0·040 056	+ 0·008 722	— 0·001 194
3	— 0·012 325	— 0·228 516	+ 0·968 300	+ 0·196 945	— 0·004 162
4	+ 0·000 296	+ 0·007 657	— 0·111 080	+ 0.977 823	— 0.002 168
5	— 0·000 075	— 0·000 022	— 0·010 148	— 0·156 936	+ 1·000 419

In the numerical evaluation of Δb_i we need only regard the terms that can contribute to the ninth decimal place of b_i, if $i = 1, \ldots 4$ and to the 12th decimal in the case of b_5, since the b_i themselves are computed, if $i = 1, \ldots 4$ and $i = 5$ to 9 and 12 decimal places respectively. The formulae for the Δb_i then reduce to:

$$\Delta b_1 = \tfrac{3}{4}\, \alpha_{11}\, (A_1)\, N_1^2 + \alpha_{21} \left[\tfrac{3}{2}(A_2) + \tfrac{1}{2}(C_2)\right] \sigma_{21}\, N_2^2 \qquad = 0{\cdot}0000\,000003$$

$$\Delta b_2 = \tfrac{3}{4}\, \alpha_{22}\, (A_2)\, N_2^2 + \tfrac{3}{4}\, \alpha_{22}\, \Sigma_\lambda\, (B_{2\lambda})\, \sigma_\lambda\, N_2^2 \qquad = 0{\cdot}0000\,000371$$

$$\Delta b_3 = \tfrac{3}{4}\, \alpha_{33}\, (A_3)\, N_3^2 \qquad = 0{\cdot}0000\,0000115$$

$$\Delta b_4 = \tfrac{3}{4}\, \alpha_{44}\, (A_4)\, N_4^2 \qquad = 0{\cdot}0000\,0000057$$

$$\Delta b_5 = -\,\alpha_{55} \left[\tfrac{9}{4}(A_{55}) + \tfrac{5}{4}(B_{55})\right] N_5^2 - \alpha_{55}\, (A_{55})\, (p_{10}'^2 + q_{10}'^2) \qquad = 0{\cdot}0000\,0000005502$$

It appears that these corrections are unimportant in the case of the first, third and fourth satellite, but in the motion of the node of the second satellite it cannot be neglected. As regards the precession, the influence of R^{IV} is pretty considerable.

On comparing the values of N_i and σ_{ij} with those of Souillart it seems legitimate to draw the conclusion that the values of Δb_i in Souillarts theory will be several times as large as the preceding Δb_i; therefore Souillart, when computing his b_i so accurately ought to have taken the influence of R^{IV} into account.

But with the accuracy with which the elements in the theory of the satellites are known at the present time, the influence of R^{IV} can be neglected, since the corrections due to it are much smaller than the inaccuracies that arise from the uncertainties of the masses; and it is not probable that in the near future the accuracy of the elements will be such as to make the above Δb_i of any importance.

The influence of R^{IV} thus appears to be negligible, except in the case of b_5. But the developments are based on the fact that from the equation:

$$\frac{dN_j}{d\tau} = \Sigma_i \alpha_{ij} \left\{ \left[\frac{\partial R_i^{IV}}{\partial Q_i} \right] \cos B_j + \left[\frac{\partial R_i^{IV}}{\partial P_i} \right] \sin B_j \right\}$$

we could conclude that $\frac{dN_j}{d\tau}$ is purely periodic, so that N_j could with sufficient approximation be regarded as a constant. It remains however questionable whether among the periodic parts there are not critical terms, the influence of which might become rather considerable on integrating the equation for N_j. A thorough investigation will show however that this is not the case and consequently the preceding results may be taken as rigorous.

Summary of results.

Before concluding we will briefly recapitulate the results, which have been arrived at. The positions of the orbital planes of the satellites, referred to Jupiter's true equator, are determined — if we take account of the secular perturbations only —, by the formulae:

$$Q_i = -\gamma_i \sqrt{G_i} \sin \theta_i = \Sigma_j \sigma_{ij} N_j \sin (b_j \tau + M_j)$$
$$P_i = \gamma_i \sqrt{G_i} \cos \theta_i = \Sigma_j \sigma_{ij} N_j \cos (b_j \tau + M_j)$$

where:

$$\log \sqrt{G_1} = 9{\cdot}900014 - 10$$
$$\log \sqrt{G_2} = 9{\cdot}950385 - 10$$
$$\log \sqrt{G_3} = 0{\cdot}001082$$
$$\log \sqrt{G_4} = 0{\cdot}062369$$

The motion of the true equator of Jupiter with respect to the fixed plane of (x, y), for which we can take the mean equator of 1900·0, is determined by:

$$\alpha = \sin \pi \sin \psi = \Sigma_j \sigma_{5j} N_j \sin (b_j \tau + M_j),$$
$$\beta = \sin \pi \cos \psi = \Sigma_j \sigma_{5j} N_j \cos (b_j \tau + M_j).$$

We have found the following values of b_i, — including the corrections resulting from R^{IV} — and of σ_{ij}:

$b_1 = +0{\cdot}0025\ 8765\ 53$	$+0{\cdot}0000\ 0000\ 03$	$= +0{\cdot}0025\ 8765\ 58$
$b_2 = +0{\cdot}0006\ 3758\ 51$	$+0{\cdot}0000\ 0003\ 71$	$= +0{\cdot}0006\ 3762\ 22$
$b_3 = +0{\cdot}0001\ 3566\ 86$	$+0{\cdot}0000\ 0000\ 11$	$= +0{\cdot}0001\ 3566\ 97$
$b_4 = +0{\cdot}0000\ 3533\ 77$	$+0{\cdot}0000\ 0000\ 06$	$= +0{\cdot}0000\ 3533\ 83$
$b_5 = +0{\cdot}0000\ 0003\ 55664$	$+0{\cdot}0000\ 0000\ 00550$	$= +0{\cdot}0000\ 0003\ 5621\ 4$

$$\sigma_{11} = + 1 \qquad \sigma_{12} = + 0\cdot02116$$
$$\sigma_{21} = - 0\cdot03222 \qquad \sigma_{22} = + 1$$
$$\sigma_{31} = - 0\cdot00122 \qquad \sigma_{32} = - 0\cdot04046$$
$$\sigma_{41} = + 0\cdot00149 \qquad \sigma_{42} = - 0\cdot00038$$
$$\sigma_{51} = + 0\cdot00138 \qquad \sigma_{52} = + 0\cdot00102$$

$$\sigma_{13} = + 0\cdot00876 \qquad \sigma_{14} = + 0\cdot00145$$
$$\sigma_{23} = + 0\cdot13677 \qquad \sigma_{24} = + 0\cdot01816$$
$$\sigma_{33} = + 1 \qquad \sigma_{34} = + 0\cdot11397$$
$$\sigma_{43} = - 0\cdot20278 \qquad \sigma_{44} = + 1$$
$$\sigma_{53} = + 0\cdot00401 \qquad \sigma_{54} = + 0\cdot00266$$

$$\sigma_{15} = + 0\cdot00048$$
$$\sigma_{25} = + 0\cdot00525$$
$$\sigma_{35} = + 0\cdot02849$$
$$\sigma_{45} = + 0\cdot15893$$
$$\sigma_{55} = + 1$$

The values of N_i, which were used for the computation of the corrections to b_i, were derived from the data given by de Sitter in "*Elements and masses*" as has been explained above (see page 50). The nodes Ω_i of the orbits of the satellites are given by:

$$\Omega_i = \theta_i + \sigma_i$$

The values σ_i are:

$$\sigma_1 = 0\cdot00000\,0000187\ \tau$$
$$\sigma_2 = 0\cdot00000\,0022119\ \tau$$
$$\sigma_3 = 0\cdot00000\,0000728\ \tau$$
$$\sigma_4 = 0\cdot00000\,0000376\ \tau$$

The unit of the time-variable τ is $1^d\cdot1221\,899$ mean solar days, while the angles are expressed in radians.

STELLINGEN.

I. De argumenten, welke SOUILLART aanvoert ter verdediging van zijn eerste integratiemethode van de differentiaalvergelijkingen voor de beweging der baanvlakken van de Jupitersatellieten, zijn niet overtuigend.

Souillart: Théorie Analytique des mouvements des Sat. de Jupiter, Memoirs R. A. Soc. Vol. XIV pp. 127—128.

II. De formules, voorkomende op pg. 56 van TISSERAND, Traité de Méc. Cél. IV, en de gevolgtrekkingen, daaruit afgeleid voor den invloed van de correctietermen op de beweging der hellingen en knoopen van de Jupitersatellieten, zijn onjuist.

III. Vergelijking van de ontwikkelingen der storingsfunctie naar de hellingen volgens HILL en LEVERRIER levert waarschijnlijk onbekende identiteiten op voor de elliptische integralen. Zoo vindt men, indien F_1 en E_1 resp. voorstellen een volledige elliptische integraal van de eerste en de tweede soort met mod. α ($0 < |\alpha| < 1$), de betrekking:

$$\frac{(1+\alpha^2)\,E_1 - (1-\alpha^2)\,F_1}{2\,(1-\alpha^2)^2} = \sum_{n=1}^{\infty} n\alpha^{2n} \int_0^{\pi} \frac{\sin^{4n}\psi\, d\psi}{\sqrt{1-\alpha^2\sin^2\psi}}$$

IV. De coefficienten van LAPLACE $L_i^{(s)}$ (i en s willekeurig, notatie van CHARLIER, „Mechanik des Himmels"), voorkomende in de ontwikkelingen der storingsfunctie kunnen worden uitgedrukt als oneindige reeksen van $L_m^{(p)}$ (m en p willekeurig), die in 't gebied, waarin de coefficienten zelf convergeeren, convergent zijn.

V. Het bewijs, dat TISSERAND geeft voor de convergentie der reeksen van de elliptische beweging naar de excentriciteiten is onvoldoende en in 't algemeen onjuist.

Tisserand, Traité de Méc. Cél. I, Chap. XVI.

VI. De wijze, waarop TISSERAND in zijn Traité de Méc. Cél. I „osculeerende elementen" definieert, verdient uit een oogpunt van zuiverheid de voorkeur boven die, door POINCARÉ gevolgd in zijn Leçons de Méc. Cél. I.

VII. De stelling, door CESARO in zijn „Lehrbuch der Algebraischen Analysis" in § 25 en nogmaals in § 35 bewezen, dat

$$D = \alpha_{rs}a_{rs} - \sum_{i,j} a_{ij}a_{is}a_{rj} \quad , \quad i = 1, 2, \ldots n,\ i \neq r,\ j = 1, 2, \ldots n,\ j \neq s,$$

waarin D is een determinant van de n'de orde, α_{rs} is de minor van het element a_{rs}, a_{ij} is het

algebraisch complement van den minor, verkregen door uit D twee horizontale rijen r en i, en de twee kolommen s en j te nemen, is in 't algemeen onjuist.

De toepassingen op blz. 17, *k* en op blz. 41, § 53 zijn, als bijzondere gevallen juist.

VIII. Het komt in de Geschiedenis der Wiskunde meermalen voor, dat een oorspronkelijke definitie ontaardt in een bepaalde eigenschap, die slechts zeer beperkt blijft, omdat nieuwe eigenschappen door de oorspronkelijke definitie ingevoerd, zoodanig op den voorgrond treden, dat ze op hun beurt definities worden en de oorspronkelijke terugbrengen tot een bepaalde eigenschap van zeer ondergeschikt belang.

IX. OSGOOD stelt in zijn „Lehrbuch der Funktionentheorie" na functies geconstrueerd te hebben, die continu zijn, differentiaalquotienten van alle orden hebben, maar toch niet in een machtreeks te ontwikkelen zijn de vraag: „Wo ist da eine geometrische oder funktionentheoretische Eigenschaft, welche wir bei unseren Funktionen nicht vermissen möchten, welche aber diesen Funktionen abginge?" Deze eigenschap kan worden aangewezen.

X. Behalve de methoden, voorkomende in ADLER, Geometrische Konstruktionen, om door middel van een hyperbool een hoek in drie deelen te verdeelen, kan ook de volgende worden genoemd:

Laat de hoek, die in drie deelen verdeeld moet worden een middelpuntshoek zijn van een cirkelsector, trek de koorde en beschouw deze als vaste basis van een veranderlijken driehoek, waarvan de basishoeken zich verhouden als 2 tot 1. Het toppunt beschrijft dan een hyperbool. Verbind het snijpunt van dezen met den cirkelboog met het middelpunt, dan is de aldus gevormde hoek $\frac{1}{3}$ van den oorspronkelijken.

XI. Bij eene elementaire behandeling van het vraagstuk van de beweging van een stoffelijk punt onder invloed van de zwaartekracht in of op een verticaal gestelden gladden cirkel verdient het aanbeveling de parabolische beweging meer op den voorgrond te plaatsen dan in de gebruikelijke leerboeken het geval is.

XII. Het is gewenscht bij de verklaring der kapillariteitsverschijnselen bij een vloeistofoppervlakte het begrip oppervlaktespanning te doen vervallen.

XIII. Het valt te betwijfelen of de bepaling van het potentiaal verschil van een punt van de lucht met de aarde door middel van de collectoren juiste resultaten oplevert.

XIV. Het is zeer in 't belang van het middelbaar en gymnasiaal onderwijs, dat van aanstaande leeraren in een der natuurwetenschappen een algemeen candidaatsexamen geëischt wordt.

PERPUSTAKAAN NASIONAL RI.